Continuity of weakly monotone Sobolev functions of variable exponent

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Abstract

Our aim in this paper is to deal with continuity properties for weakly monotone Sobolev functions of variable exponent.

1 Introduction

This paper deals with continuity properties of weakly monotone Sobolev functions. We begin with the definition of weakly monotone functions. Let D be an open set in the n-dimensional Euclidean space $\mathbf{R}^n$ ($n \geq 2$). A function u in the Sobolev space $W^{1,q}_{loc}(D)$ is said to be weakly monotone in D (in the sense of Manfredi [12]), if for every relatively compact subdomain G of D and for every pair of constants $k \leq K$ such that

$$(k-u)^+ \qquad \text{and} \qquad (u-K)^+ \in W_0^{1,q}(G),$$

we have

$$k \leq u(x) \leq K \qquad \text{for a.e. } x \in G,$$

where $v^+(x) = \max\{v(x), 0\}$. If a weakly monotone Sobolev function is continuous, then it is monotone in the sense of Lebesgue [11]. For monotone functions, see Koskela-Manfredi-Villamor [9], Manfredi-Villamor [13, 14], the second author [17], Villamor-Li [20] and Vuorinen [21, 22].

Following Kováčik and Rákosník [10], we consider a positive continuous function $p(\cdot) : D \to (1, \infty)$ and the Sobolev space $W^{1,p(\cdot)}(D)$ of all functions u whose first (weak) derivatives belong to $L^{p(\cdot)}(D)$. In this paper we consider the function $p(\cdot)$ satisfying

$$|p(x) - p(y)| \leq \frac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/2$, for $a \geq 0$ and $b \geq 0$.

2000 Mathematics Subject Classification : Primary 30C65, 31B15 (46E35)

Key words and phrases : weakly monotone Sobolev functions of variable exponent, 0-Hölder continuous, capacity, tangential boundary limits

Our first aim is to discuss the continuity for weakly monotone functions u in the Sobolev space $W^{1,p(\cdot)}(D)$. For the properties of Sobolev spaces of variable exponent, we refer the reader to the papers by Diening [2], Edmunds-Rákosník [3], Kováčik-Rákosník [10] and Růžička [19].

We know that if $p(x) \geq n$ for all $x \in D$, then all weakly monotone functions in $W^{1,p(\cdot)}(D)$ are continuous in D (see Manfredi [12] and Manfredi-Villamor [13]). We show that u is continuous at $x_0 \in D$ when $p(\cdot)$ is of the form

$$p(x) = n - \frac{a \log(\log(1/|x - x_0|))}{\log(1/|x - x_0|)} \qquad (p(x_0) = n)$$

for $x \in B(x_0, r_0)$, where $0 < r_0 < 1/2$ and $a \leq 1$.

Our second aim is to prove the existence of boundary limits of weakly monotone Sobolev functions on the unit ball B, when $p(\cdot)$ satisfies the inequality

$$\left| p(x) - \left\{ n + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \leq \frac{b}{\log(e/\rho(x))}$$

for $a \geq 0$ and $b \geq 0$, where $\rho(x) = 1 - |x|$ denotes the distance of x from the boundary ∂B. Continuity of Sobolev functions has been obtained by Harjulehto-Hästö [7] and the authors [4]. Of course, our results extend the non-variable case studied in [17].

2 Weakly monotone Sobolev functions

Throughout this paper, let C denote various constants independent of the variables in question.

We use the notation $B(x, r)$ to denote the open ball centered at x of radius r. If u is a weakly monotone Sobolev function on D and $q > n - 1$, then

$$|u(x) - u(x')|^q \leq C r^{q-n} \int_{A(y,2r)} |\nabla u(z)|^q dz \tag{1}$$

for almost every $x, x' \in B(y, r)$, whenever $B(y, 2r) \subset D$ (see [12, Theorem 1]) and $A(y, 2r) = B(y, 2r) \setminus B(y, r)$. If we define $u^*(x)$ by

$$u^*(x) = \limsup_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} u(y) dy,$$

then we see that u^* satisfies (1) for all $x, x' \in B(y, r)$. Note here that u^* is a quasicontinuous representative of u and it is locally bounded on D. Hereafter, we identify u with u^*.

EXAMPLE 2.1. Let $1 < q < \infty$ and $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping satisfying the following assumptions for some measurable function α and constant β such that $0 < \alpha(x) \leq \beta < \infty$ for a.e. $x \in \mathbf{R}^n$:

(i) the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbf{R}^n$,

(ii) the mapping $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbf{R}^n$,

(iii) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha(x)|\xi|^q \quad$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$,

(iv) $|\mathcal{A}(x, \xi)| \leq \beta|\xi|^{q-1} \quad$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$.

Then a weak solution of the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u(x)) = 0 \tag{2}$$

in an open set D is weakly monotone (see [9, Lemma 2.7]). In the special case $\alpha(x) \geq \alpha > 0$, according to the well-known book by Heinonen-Kilpeläinen-Martio [8], a weak solution of (2) is monotone in the sense of Lebesgue.

3 Continuity of weakly monotone functions

For an open set G in $\mathbf{R}^n$, define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leq 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$. We denote by $W^{1,p(\cdot)}(G)$ the space of all functions $u \in L^{p(\cdot)}(G)$ whose first (weak) derivatives belong to $L^{p(\cdot)}(G)$. We define the conjugate exponent function $p'(\cdot)$ to satisfy $1/p(x) + 1/p'(x) = 1$.

Let $B(x, r)$ be the open ball centered at x and radius $r > 0$, and let $B = B(0, 1)$. Consider a positive continuous function $p(\cdot)$ on $[0, 1]$ such that

$$\left| p(r) - \left\{ n - \frac{a \log(e + \log(1/r))}{\log(e/r)} \right\} \right| \leq \frac{b}{\log(e/r)} \qquad (p(0) = n)$$

for $a \geq 0$ and $b \geq 0$.

Our aim in this section is to prove that if $a \leq 1$, then functions in $W^{1,p(\cdot)}(B)$ are continuous at the origin, in spite of the fact that $\inf_{x \in B} p(x) < n$. For this purpose, we prepare the following result.

Lemma 3.1. *Let $p(x) = p(|x|)$ for $x \in B$. Let u be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $a < 1$, then*

$$|u(x) - u(0)|^n \leq C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy,$$

and if $a = 1$, then

$$|u(x) - u(0)|^n \leq C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy$$

whenever $|x| < r < 1/4$, *where* $R = \sqrt{r}$ *when* $a < 1$ *and* $R = e^{-\sqrt{\log(1/r)}}$ *when* $a = 1$.

Proof. Let u be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. Set $p_1(r) = p(r)/q$, where $n-1 < q < n$. Then, as in (1), we apply Sobolev's theorem on the sphere $S(0,r)$ to establish

$$|u(x) - u(0)|^q \le Cr^{q-(n-1)} \int_{S(0,r)} |\nabla u(y)|^q dS(y)$$

for $|x| < r$. By Hölder's inequality we have

$$\begin{aligned} & |u(x) - u(0)|^q \\ \le\ & Cr^{q-(n-1)} \left(\int_{S(0,r)} dS(y) \right)^{1/p_1'(r)} \left(\int_{S(0,r)} |\nabla u(y)|^{qp_1(r)} dS(y) \right)^{1/p_1(r)} \\ \le\ & Cr^{q-(n-1)/p_1(r)} \left(\int_{S(0,r)} |\nabla u(y)|^{p(r)} dS(y) \right)^{1/p_1(r)}, \end{aligned}$$

which yields

$$|u(x) - u(0)|^{p(r)} \le Cr(\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y)$$

for $|x| < r$. Since u is bounded on $B(0,1/2)$, we see that

$$|u(x) - u(0)|^n \le Cr(\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y).$$

Hence, by dividing both sides by $r(\log(1/r))^a$ and integrating them on the interval (r,R), we obtain

$$|u(x) - u(0)|^n \le C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a < 1$$

and

$$|u(x) - u(0)|^n \le C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a = 1$$

whenever $|x| < r < 1/4$. □

Lemma 3.1 yields the following result.

Theorem 3.2. *Let* u *be a weakly monotone Sobolev function in* $W^{1,p(\cdot)}(B)$. *If* $a < 1$, *then* u *is continuous at the origin and it satisfies*

$$\lim_{x \to 0} (\log(1/|x|))^{(1-a)/n} |u(x) - u(0)| = 0;$$

if $a = 1$, *then*

$$\lim_{x \to 0} (\log(\log(1/|x|)))^{1/n} |u(x) - u(0)| = 0.$$

REMARK 3.3. Consider the function

$$u(x) = \frac{x_n}{|x|}$$

for $x = (x_1, ..., x_n)$. If we define $u(0) = 0$, then u is a weakly monotone quasi-continuous representative in $\mathbf{R}^n$. Note that u is not continuous at 0 and if $a > 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty;$$

if $a \leq 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx = \infty.$$

This shows that continuity result in Theorem 3.2 is good as to the size of a.

REMARK 3.4. Let φ be a nonnegative continuous function on the interval $[0, r_0]$ such that

(i) $\varphi(0) = 0$;

(ii) $\varphi'(t) \geq 0$ for $0 < t < r_0$;

(iii) $\varphi''(t) \leq 0$ for $0 < t < r_0$.

Then note that

$$\varphi(s + t) \leq \varphi(s) + \varphi(t) \tag{3}$$

for $s, t \geq 0$ and $s + t \leq r_0$. Consider

$$\varphi(r) = \frac{\log(\log(1/r))}{\log(1/r)}, \quad \frac{1}{\log(1/r)}$$

for $0 < r \leq r_0$; set $\varphi(r) = \varphi(r_0)$ for $r > r_0$. Then we can find $r_0 > 0$ such that φ satisfies (i) - (iii) on $[0, r_0]$, and hence (3) holds for all $s \geq 0$ and $t \geq 0$. Hence if we set

$$p(r) = n + \frac{a \log(e + \log(1/r))}{\log(e/r)} + \frac{b}{\log(e/r)},$$

then we can find $c > 0$ and $r_0 > 0$ such that

$$|p(s) - p(t)| \leq \frac{|a| \log(\log(1/|s - t|))}{\log(1/|s - t|)} + \frac{c}{\log(1/|s - t|)}$$

whenever $|s - t| < r_0$.

4 0-Hölder continuity of continuous Sobolev functions

Consider a positive continuous function $p(\cdot)$ on the unit ball B such that $p(x) > p_0$ and

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

for all $x \in B$, where $1 < p_0 < \infty$ and $\rho(x) = 1 - |x|$ denotes the distance of x from the boundary ∂B. Then note that

$$\begin{aligned} p'(x) - p'(0) &= -\frac{p(x) - p(0)}{(p(x)-1)(p(0)-1)} \\ &= -\frac{p(x)-p(0)}{(p(0)-1)^2} + \frac{(p(x)-p(0))^2}{(p(x)-1)(p(0)-1)^2}. \end{aligned}$$

Hence we have the following result.

Lemma 4.1. *There exist positive constants r_0 and C such that*

$$|p'(x) - \{p_0' - \omega(\rho(x))\}| \le C/\log(1/\rho(x))$$

for $x \in B$, where $p_0' = p_0/(p_0 - 1)$ and $\omega(t) = (a/(p_0-1)^2)\log(\log(1/t))/\log(1/t)$ for $0 < r \le r_0 < 1/e$; set $\omega(t) = \omega(r_0)$ for $r > r_0$.

In view of Sobolev's theorem, we see that all functions $u \in W^{1,p(\cdot)}(B)$ are continuous in B. In what follows we discuss the 0-Hölder continuity of u. Before doing so, we need the following result.

Lemma 4.2. *Let $p_0 = n$ and let u be a continuous Sobolev function in $W^{1,p(\cdot)}(B)$ such that $\| |\nabla u| \|_{p(\cdot)} \le 1$. If $a > n - 1$, then*

$$\int_{B \cap B(x,r)} |x-y|^{1-n} |\nabla u(y)| \le C(\log(1/r))^{-A},$$

where $A = (a - n + 1)/n$.

Proof. Let $f(y) = |\nabla u(y)|$ for $y \in B$ and $f = 0$ outside B. For $0 < \mu < 1$, we have

$$\begin{aligned} &\int_{B(x,r)} |x-y|^{1-n} f(y)dy \\ \le\ & \mu \left\{ \int_{B(x,r)} (|x-y|^{1-n}/\mu)^{p'(y)} dy + \int_{B(x,r)} f(y)^{p(y)} dy \right\} \\ \le\ & \mu \left\{ \mu^{-n/(n-1)} \int_{B(x,r)} |x-y|^{(1-n)p'(y)} dy + 1 \right\}. \end{aligned}$$

Applying polar coordinates, we have

$$\begin{aligned}\int_{B(x,r)} |x-y|^{(1-n)p'(y)}dy &\le C\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(1-n)(n'-\omega_0(t))+n-1}dt \\ &= C\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt,\end{aligned}$$

where $\omega_0(t) = \omega(t) - C/\log(1/t)$. If $r \le \rho(x)/2$ and $|\rho(x)-t| < \rho(x)/2$, then

$$\omega_0(t) \ge \omega(r) - C/\log(1/r),$$

so that

$$\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt \le C(\log(1/r))^{1-a/(n-1)}.$$

If $r > \rho(x)/2$, then $|t| < 3|\rho(x)-t|$ when $|\rho(x)-t| \ge \rho(x)/2$. Hence, in this case, we obtain

$$\begin{aligned}&\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt \\ \le\ & \int_{\{t:|t-\rho(x)|<\rho(x)/2\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt + C\int_{\{t:|t|<3r\}} |t|^{(n-1)\omega_0(t)-1}dt \\ \le\ & C(\log(1/r))^{1-a/(n-1)},\end{aligned}$$

so that

$$\int_{B(x,r)} |x-y|^{(1-n)p'(y)}dy \le C(\log(1/r))^{1-a/(n-1)}.$$

Consequently it follows that

$$\int_{B(x,r)} |x-y|^{1-n}f(y)dy \le \mu\left(C\mu^{-n/(n-1)}(\log(1/r))^{1-a/(n-1)} + 1\right).$$

Now, letting $\mu^{-n/(n-1)}(\log(1/r))^{1-a/(n-1)} = 1$, we establish

$$\int_{B(x,r)} |x-y|^{1-n}f(y)dy \le C(\log(1/r))^{(n-1-a)/n}.$$

□

Now we are ready to show the 0-Hölder continuity of Sobolev functions in $W^{1,p(\cdot)}(B)$.

THEOREM 4.3. *Let $p_0 = n$ and u be a continuous Sobolev function in $W^{1,p(\cdot)}(B)$ such that $\||\nabla u|\|_{p(\cdot)} \le 1$. If $a > n-1$, then*

$$|u(x) - u(y)| \le C(\log(1/|x-y|))^{-A}$$

whenever x, $y \in B$ and $|x-y| < 1/2$.

PROOF. Let x, $y \in B$ and $r = |x-y| \le \rho(x)$. Then we see from Lemma 4.2 that

$$|u(x) - u(y)| \le C \int_{B(x,r)} |x-z|^{1-n} |\nabla u(z)| dz \le C(\log(1/r))^{-A}.$$

If $r = |x-y| < 1/2$, $\rho(x) < r$ and $\rho(y) < r$, then we take $x_r = (1-r)x/|x|$ and $y_r = (1-r)y/|y|$ to establish

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(x_r)| + |u(x_r) - u(y_r)| + |u(y_r) - u(y)| \\ &\le C(\log(1/r))^{-A}, \end{aligned}$$

which proves the assertion. □

REMARK 4.4. Let $p(\cdot)$ be as above, and consider the function

$$u(x) = [\log(e + \log(1/|x-\xi|))]^{\delta},$$

where $\xi \in \partial B$ and $0 < \delta < (n-1)/n$. We see readily that $u(\xi) = \infty$ and it is monotone in B. Further, if $a \le n-1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty,$$

so that Theorem 4.3 does not hold for $a \le n-1$.

5 Tangential boundary limits of weakly monotone Sobolev functions

Let G be a bounded open set in $\mathbf{R}^n$. Consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ satisfying

(p1) $p_-(G) = \inf_G p(x) > 1$ and $p_+(G) = \sup_G p(x) < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{b}{\log(1/|x-y|)}$ whenever $|x-y| < 1/e$, where $a \ge 0$ and $b \ge 0$.

For $E \subset G$, we define the relative $p(\cdot)$-capacity by

$$C_{p(\cdot)}(E; G) = \inf \int_G f(y)^{p(y)} dy,$$

where the infimum is taken over all nonnegative functions $f \in L^{p(\cdot)}(G)$ such that

$$\int_G |x-y|^{1-n} f(y) dy \ge 1 \qquad \text{for every } x \in E.$$

From now on we collect fundamental properties for our capacity (see Meyers [15], Adams-Hedberg [1] and the authors [6]).

LEMMA 5.1. *For $E \subset G$, $C_{p(\cdot)}(E;G)=0$ if and only if there exists a nonnegative function $f \in L^{p(\cdot)}(G)$ such that*

$$\int_G |x-y|^{1-n} f(y)dy = \infty \qquad \text{for every } x \in E.$$

For $0<r<1/2$, set

$$h(r;x) = \begin{cases} r^{n-p(x)}(\log(1/r))^a & \text{when } p(x)<n, \\ (\log(1/r))^{a-(n-1)} & \text{when } p(x)=n \text{ and } a<n-1, \\ (\log(\log(1/r)))^{-a} & \text{when } p(x)=n \text{ and } a=n-1, \\ 1 & \text{when } p(x)=n \text{ and } a>n-1. \end{cases}$$

LEMMA 5.2. *Suppose $p(x_0) \le n$ and $a \le n-1$. If $B(x_0,r) \subset G$ and $0<r<1/2$, then*

$$C_{\alpha,p(\cdot)}(B(x_0,r);G) \le Ch(r;x_0).$$

LEMMA 5.3. *If f is a nonnegative measurable function on G with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r\to 0+} h(r;x)^{-1} \int_{B(x,r)} f(y)^{p(y)} dy = 0$$

holds for all x except in a set $E \subset G$ with $C_{p(\cdot)}(E;G)=0$.

Let $p(\cdot)$ be as in Section 4; that is, we assume that $p(x)>n$ and

$$\left| p(x) - \left\{ n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))} \tag{4}$$

for $x \in B$, where $a \ge 0$ and $b>0$. Then $p_1(x) \le p(x) \le p_2(x)$ for $x \in B$, where

$$\begin{aligned} p_1(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}, \\ p_2(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} + \frac{b}{\log(e/\rho(x))}. \end{aligned}$$

For simplicity, set

$$p(x) = p_1(x) = p_2(x) = n$$

outside B. Then we can find $b' > b$ such that for $i=1,2$

$$\begin{aligned} |p_i(x)-p_i(y)| &\le \frac{a\log(e+\log(1/|x-y|))}{\log(e/|x-y|)} + \frac{b}{\log(e/|x-y|)} \\ &\le \frac{a\log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b'}{\log(1/|x-y|)} \end{aligned}$$

whenever $|x-y|$ is small enough, say $|x-y|<r_0<1/e$.

Since G has finite measure, we find a constant $K>0$ such that

$$C_{\alpha,p(\cdot)}(E;G) \le K C_{\alpha,p_2(\cdot)}(E;G) \tag{5}$$

whenever $E \subset G$. Hence, in view of Lemma 5.2, we obtain

$$C_{\alpha,p(\cdot)}(B(x_0,r);2B) \le Ch(r;x_0) \tag{6}$$

for $x_0 \in \partial B$, where $2B = B(0,2)$.

COROLLARY 5.4. *If f is a nonnegative measurable function on $2B$ with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r\to 0+} h(r;x)^{-1} \int_{B(x,r)} f(y)^{p(y)}dy = 0$$

holds for all $x \in \partial B$ except in a set $E \subset \partial B$ with $C_{p(\cdot)}(E;2B)=0$.

If u is a weakly monotone function in $W^{1,p(\cdot)}(B)$, then, since $p(x)>n$ for $x \in B$ by our assumption, we see that u is continuous in B and hence monotone in B in the sense of Lebesgue. We now show the existence of tangential boundary limits of monotone Sobolev functions u in B when $a \le n-1$.

For $\xi \in \partial B$, $\gamma \ge 1$ and $c>0$, set

$$T_\gamma(\xi,c) = \{x \in B : |x-\xi|^\gamma < c\rho(x)\}.$$

THEOREM 5.5. *Let $p(\cdot)$ be a positive continuous function on $2B$ such that $p(x) \ge n$ for $x \in 2B$ and*

$$\left| p(x) - \left\{ n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

for $x \in B$, where $a \ge 0$ and $b>0$. If u is a monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Lebesgue), then there exists a set $E \subset \partial B$ such that

(i) $C_{p(\cdot)}(E;B(0,2)) = 0$;

(ii) *if $\xi \in \partial B \setminus E$, then $u(x)$ has a finite limit as $x \to \xi$ along the sets $T_\gamma(\xi,c)$.*

If $a>n-1$, then the above function u has a continuous extension on $\overline{B} = B \cup \partial B$ in view of Theorem 4.3, and hence the exceptional set E can be taken as the empty set.

To prove Theorem 5.5, we may assume that

$$p(x) = n + \frac{a\log(e+\log(e/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}$$

for $x \in B$.

We need the following two results. The first one follows from inequality (1) (see e.g. [9] and [5]).

LEMMA 5.6. *Let u be a monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $\xi \in \partial B$, $x \in B$ and $n-1 < q < n$, then*

$$|u(x) - u(\tilde{x})|^q \le C(\log(2r/\rho(x)))^{q-1} \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy,$$

where $\tilde{x} = (1-r)\xi$, $r = |\xi - x|$ and $E(x) = \cup_{y \in \overline{x\tilde{x}}} B(y, \rho(y)/2)$ with $\overline{x\tilde{x}} = \{tx + (1-t)\tilde{x} : 0 < t < 1\}$.

LEMMA 5.7. *Let u be a monotone Sobolev function in $W^{1,p(\cdot)}(B)$. Let $\xi \in \partial B$ and $a \ge 0$. Suppose*

$$(\log(1/r))^{n-1-a} \int_{B \cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy \le 1.$$

If $x \in T_\gamma(\xi, c)$, $\tilde{x} = (1-r)\xi$ and $r = |\xi - x|$, then

$$|u(x) - u(\tilde{x})|^n \le C(\log(1/r))^{n-1-a} \int_{B \cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy.$$

PROOF. First note that

$$\rho(y) \ge C(\rho(x) + |x - y|) \qquad \text{for } y \in E(x).$$

Take q such that $n-1 < q < n$; when $a > 0$, assume further that $a > (n-q)/q$. Set $p_1(x) = p(x)/q$. Then we have for $\mu > 0$

$$\begin{aligned} \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy &\le \mu \left\{ \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + \int_{E(x)} |\nabla u(y)|^{qp_1(y)} dy \right\} \\ &\le \mu \left\{ \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + F \right\}, \end{aligned}$$

where $F = \int_{E(x)} |\nabla u(y)|^{p(y)} dy$. Note from Lemma 4.1 that

$$|p_1'(y) - \{n/(n-q) - \omega(\rho(y))\}| \le C/\log(1/\rho(y))$$

for $y \in E(x)$, where $\omega(t) = (aq^2/(n-q)^2) \log(\log(1/t))/\log(1/t)$. Hence

$$n/(n-q) - \omega_1(\rho(y)) \le p_1'(y) \le n/(n-q) - \omega_2(\rho(y)),$$

where $\omega_1(t) = \omega(t) + C/\log(1/t)$ and $\omega_2(t) = \omega(t) - C/\log(1/t)$. Suppose

$$(\log(1/r))^{-1+aq/(n-q)} F > 1.$$

Since $p_1'(y) \le n/(n-q)$, we have for $0 < \mu < 1$,

$$\begin{aligned}
& \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \\
\le\ & C\mu^{-n/(n-q)} \int_{E(x)} (\rho(x) + |x-y|)^{(q-n)(n/(n-q)-\omega_2(\rho(y)))} dy \\
\le\ & C\mu^{-n/(n-q)} \int_0^{2r} (\rho(x)+t)^{-n} (\log(1/(\rho(x)+t)))^{-aq/(n-q)} t^{n-1} dt \\
\le\ & C\mu^{-n/(n-q)} (\log(1/r))^{1-aq/(n-q)}
\end{aligned}$$

whenever $x \in T_\gamma(\xi, c)$. Considering

$$\mu^{-n/(n-q)} (\log(1/r))^{1-aq/(n-q)} = F,$$

we obtain

$$\begin{aligned}
\int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \ &\le\ C \left\{ (\log(1/r))^{-1+aq/(n-q)} F \right\}^{-(n-q)/n} F \\
&=\ C \left\{ (\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy \right\}^{q/n}.
\end{aligned}$$

Consequently it follows from Lemma 5.6 that

$$|u(x) - u(\tilde{x})|^n \le C(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$.

Next consider the case when $(\log(1/r))^{-1+aq/(n-q)} F \le 1$. Set $p^+ = \sup_{B \cap B(\xi,2r)} p(y)$ and and $p_1^+ = \sup_{B \cap B(\xi,2r)} p_1(y) = p^+/q$. For $\mu > 1$, we apply the above considerations to obtain

$$\begin{aligned}
\int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \ &\le\ C\mu^{-(p_1^+)'} \int_{E(x)} (\rho(x) + |x-y|)^{(q-n)(n/(n-q)-\omega_2(\rho(y)))} dy \\
&\le\ C\mu^{-(p_1^+)'} (\log(1/r))^{1-aq/(n-q)}.
\end{aligned}$$

If we take μ satisfying $\mu^{-(p_1^+)'} (\log(1/r))^{1-aq/(n-q)} = F$, then we have

$$\int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \ \le\ C \left\{ (\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy \right\}^{1/p_1^+}.$$

Since $(\log(1/r))^{\omega(r)}$ is bounded above for small $r > 0$, Lemma 5.6 yields

$$|u(x) - u(\tilde{x})|^{p^+} \le C(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$, which proves the required assertion. □

Proof of Theorem 5.5. Consider $E = E_1 \cup E_2$, where

$$E_1 = \{\xi \in \partial B : \int_B |\xi - y|^{1-n} |\nabla u(y)| dy = \infty\}$$

and

$$E_2 = \{\xi \in \partial B : \limsup_{r \to 0+} (\log(1/r))^{n-1-a} \int_{B(\xi,r)} |\nabla u(y)|^{p(y)} dy > 0\}.$$

We see from Lemma 5.1 and Corollary 5.4 that $E = E_1 \cup E_2$ is of $C_{p(\cdot)}$-capacity zero. If $\xi \notin E_1$, then we can find a line L along which u has a finite limit ℓ. In view of inequality (1), we see that u has a radial limit ℓ at ξ, that is, $u(r\xi)$ tends to ℓ as $r \to 1 - 0$. Now we insist from Lemma 5.7 that if $\xi \in \partial B \setminus E$, then $u(x)$ tends to ℓ as x tends to ξ along the sets $T_\gamma(\xi, c)$. □

Remark 5.8. If $a > n - 1$, then we do not need the monotonicity in Theorem 5.5, because of Theorem 4.3.

Finally we show the nontangential limit result for weakly monotone Sobolev functions. Recall that a quasicontinuous representative is locally bounded.

Theorem 5.9. *Let $p(\cdot)$ be a positive continuous function on B such that*

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))},$$

where $-\infty < a < \infty$, $b \ge 0$ and $n-1 < p_0 \le n$. If u is a weakly monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Manfredi), then there exists a set $E \subset \partial B$ such that

(i) $C_{p(\cdot)}(E; B(0,2)) = 0$;

(ii) *if $\xi \in \partial B \setminus E$, then $u(x)$ has a finite limit as $x \to \xi$ along the sets $T_1(\xi, c)$.*

To prove this, we need the following lemma instead of Lemma 5.7, which can be proved by use of (1) with $q = p_- = \inf_{z \in B(x,\rho(x)/2)} p(z)$.

Lemma 5.10. *Let p and u be as in Theorem 5.9. If $y \in B(x,r)$ with $r = \rho(x)/4$, then*

$$|u(x) - u(y)|^{p_-} \le C r^{p_0 - n} (\log(1/r))^{-a} \left(r^n + \int_{B(x,2r)} |\nabla u(z)|^{p(z)} dz \right).$$

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