RIGIDITY OF THE CANONICAL ISOMETRIC IMBEDDING OF THE QUATERNION PROJECTIVE PLANE $P^2(\boldsymbol{H})$

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ABSTRACT. In this paper, we investigate isometric immersions of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ and prove that the canonical isometric imbedding $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, which is defined in Kobayashi [11], is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set $U(\subset P^2(\boldsymbol{H}))$ into $\boldsymbol{R}^{14}$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{14}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

1. INTRODUCTION

In our previous paper [8], we proved the rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(\boldsymbol{Cay})$. The purpose of this paper is to investigate a similar problem for (local) isometric immersions of the quaternion projective plane $P^2(\boldsymbol{H})$. As we have proved in [7], any open set of the quaternion projective plane $P^2(\boldsymbol{H})$ cannot be isometrically immersed into $\boldsymbol{R}^{13}$. On the other hand, there is an isometric immersion $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$, which is called the canonical isometric imbedding of $P^2(\boldsymbol{H})$ (see Kobayashi [11]). Therefore, it follows that $\boldsymbol{R}^{14}$ is the least dimensional euclidean space into which $P^2(\boldsymbol{H})$ can be (locally) isometrically immersed.

In the present paper, we will show that the canonical isometric imbedding $\boldsymbol{f}_0$ is rigid in the following strongest sense:

Theorem 1. *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$. Then, for any isometric immersion $\boldsymbol{f}_1$ defined on a connected open set U of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, there exists a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

The proof of this theorem will be given by solving the Gauss equation associated with the isometric imbeddings (immersions) of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ in the same line of [8] (see

Date: April 8, 2004.

2000 *Mathematics Subject Classification.* 17B20, 53B25, 53C24, 53C35.

Key words and phrases. Curvature invariant, isometric immersion, quaternion projective plane, rigidity, root space decomposition.

Theorem 7). We use the same notations and terminology as those of the previous papers [6], [7] and [8].

2. THE QUATERNION PROJECTIVE PLANE $P^2(\boldsymbol{H})$

In this section we review the structure of the quaternion projective plane $P^2(\boldsymbol{H})$ and prepare several formulas concerning the bracket operation.

As is well-known, $P^2(\boldsymbol{H})$ can be represented by $P^2(\boldsymbol{H}) = G/K$, where $G = Sp(3)$ and $K = Sp(2) \times Sp(1)$. Let g (resp. k) be the Lie algebra of G (resp. K) and let $\mathrm{g} = \mathrm{k} + \mathrm{m}$ be the canonical decomposition of g associated with the symmetric pair (G, K). We denote by $(\ ,\ )$ the inner product of g given by the (-1)-multiple of the Killing form of g. As usual, we can identify m with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G-invariant Riemannian metric g of G/K satisfies

$$g_o(X, Y) = (X, Y), \quad X, Y \in \mathrm{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type $(1, 3)$ is given by

$$R_o(X, Y)Z = -\big[[X, Y], Z\big], \qquad \forall X, Y, Z \in \mathrm{m}.$$

We now take a maximal abelian subspace a of m and fix it in the following discussions. We note that since $\operatorname{rank}(P^2(\boldsymbol{H})) = 1$, we have $\dim \mathrm{a} = 1$.

For each element $\lambda \in \mathrm{a}$ we define two subspaces $\mathrm{k}(\lambda)$ $(\subset \mathrm{k})$ and $\mathrm{m}(\lambda)$ $(\subset \mathrm{m})$ by

$$\mathrm{k}(\lambda) = \Big\{X \in \mathrm{k} \,\big|\, [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathrm{a}\Big\},$$
$$\mathrm{m}(\lambda) = \Big\{Y \in \mathrm{m} \,\big|\, [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathrm{a}\Big\}.$$

Let Σ be the set of all non-zero restricted roots. (An element $\lambda \in \mathrm{a}$ is called a *restricted root* if $\mathrm{m}(\lambda) \neq 0$.) As is known, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ. For each integer i we set $\mathrm{k}_i = \mathrm{k}(|i|\mu)$, $\mathrm{m}_i = \mathrm{m}(|i|\mu)$ $(|i| \leq 2)$, $\mathrm{k}_i = \mathrm{m}_i = 0$ $(|i| > 2)$. Then, we have $\mathrm{m}_0 = \mathrm{a} = \boldsymbol{R}\mu$ and

$$\mathrm{k} = \mathrm{k}_0 + \mathrm{k}_1 + \mathrm{k}_2 \ (\textit{orthogonal direct sum}),$$
$$\mathrm{m} = \mathrm{m}_0 + \mathrm{m}_1 + \mathrm{m}_2 \ (\textit{orthogonal direct sum}).$$

The dimensions of the factors are given by $\dim \mathrm{k}_0 = 6$, $\dim \mathrm{k}_1 = \dim \mathrm{m}_1 = 4$ and $\dim \mathrm{k}_2 = \dim \mathrm{m}_2 = 3$ (precisely, see [7]).

We now show several formulas concerning the bracket operation of g. By the definition of the subspaces k_i and m_i we easily have

$$\big[\mathrm{k}_i, \mathrm{k}_j\big] \subset \mathrm{k}_{i+j} + \mathrm{k}_{i-j}, \quad \big[\mathrm{m}_i, \mathrm{m}_j\big] \subset \mathrm{k}_{i+j} + \mathrm{k}_{i-j}, \quad \big[\mathrm{k}_i, \mathrm{m}_j\big] \subset \mathrm{m}_{i+j} + \mathrm{m}_{i-j}. \tag{2.1}$$

Moreover, we have

Proposition 2. *Let Y_0, $Y_0' \in \mathrm{a}+\mathrm{m}_2$, Y_1, $Y_1' \in \mathrm{m}_1$. Then:*

$$\big[Y_i, \big[Y_i, Y_j'\big]\big] = -\,(1+3\delta_{ij})\,(\mu,\mu)\,\big\{\big(Y_i,Y_i\big)Y_j' - \big(Y_i,Y_j'\big)Y_i\big\}, \quad (i,j=0,1), \tag{2.2}$$

$$\big[Y_i, \big[Y_i', Y_j\big]\big] + \big[Y_i', \big[Y_i, Y_j\big]\big] = -2\big(\mu,\mu\big)\big(Y_i,Y_i'\big)Y_j, \quad (i,j=0,1,\ i\neq j), \tag{2.3}$$

$$\big[Y_i, \big[Y_i, X_1\big]\big] = -\big(\mu,\mu\big)\big(Y_i,Y_i\big)X_1, \quad \forall X_1 \in \mathrm{k}_1 \quad (i=0,1), \tag{2.4}$$

where δ_{ij} denotes the Kronecker delta.

Proof. We first prove (2.2). Assume that $i=j$ and $Y_i \neq 0$. Set $Y_i'' = Y_i' - \big(Y_i',Y_i\big)/\big(Y_i,Y_i\big)\cdot Y_i$. Then, we know that $\big(Y_i,Y_i''\big)=0$ and that $Y_i'' \in \mathrm{a}+\mathrm{m}_2$ if $i=0$ and $Y_i'' \in \mathrm{m}_1$ if $i=1$. Hence, by Proposition 10 of [7], we have $\big[Y_i,\big[Y_i,Y_i''\big]\big] = -4\big(\mu,\mu\big)\big(Y_i,Y_i\big)Y_i''$. Therefore, we can easily obtain (2.2) in the case $i=j$. In the case $i\neq j$, (2.2) directly follows from Proposition 10 of [7].

We next prove (2.3). Since $i \neq j$, it follows that $\big(Y_i,Y_j\big) = \big(Y_i',Y_j\big) = 0$. Hence, by (2.2) we have $\big[Y_i+Y_i',\big[Y_i+Y_i',Y_j\big]\big] = -\big(\mu,\mu\big)\big(Y_i+Y_i',Y_i+Y_i'\big)Y_j$. This, together with $\big[Y_i,\big[Y_i,Y_j\big]\big] = -\big(\mu,\mu\big)\big(Y_i,Y_i\big)Y_j$ and $\big[Y_i',\big[Y_i',Y_j\big]\big] = -\big(\mu,\mu\big)\big(Y_i',Y_i'\big)Y_j$, proves (2.3).

We finally prove (2.4). We note that $\big[Y_1,\mathrm{a}+\mathrm{m}_2\big] = \mathrm{k}_1$ holds for any $Y_1 \in \mathrm{m}_1\,(\neq 0)$. In fact, it is easy to see $\big[Y_1,\mathrm{a}+\mathrm{m}_2\big] \subset \mathrm{k}_1$ (see (2.1)). Moreover, the map $\mathrm{a}+\mathrm{m}_2 \ni Y_0' \longmapsto \big[Y_1,Y_0'\big] \in \mathrm{k}_1$ is bijective, because $\big[Y_1,Y_0'\big] \neq 0$ if $Y_0' \in \mathrm{a}+\mathrm{m}_2\,(Y_0'\neq 0)$ (recall that $\operatorname{rank}(P^2(\boldsymbol{H}))=1$) and because $\dim(\mathrm{a}+\mathrm{m}_2) = \dim \mathrm{k}_1$. Let $X_1 \in \mathrm{k}_1$. Then, by $\big[Y_1,\mathrm{a}+\mathrm{m}_2\big] = \mathrm{k}_1$ we can take an element $Y_0' \in \mathrm{a}+\mathrm{m}_2$ such that $\big[Y_1,Y_0'\big] = X_1$. Now, applying $\operatorname{ad} Y_1$ to the equality $\big[Y_1,\big[Y_1,Y_0'\big]\big] = -\big(\mu,\mu\big)\big(Y_1,Y_1\big)Y_0'$ (see (2.2)), we have $\big[Y_1,\big[Y_1,X_1\big]\big] = -\big(\mu,\mu\big)\big(Y_1,Y_1\big)X_1$, proving (2.4) for the case $i=1$. Similarly, we can prove (2.4) for the case $i=0$. Λ

Let Y_0, $Y_0' \in \mathrm{a}+\mathrm{m}_2$. Define a linear mapping $L(Y_0,Y_0')$ of m_1 to m by

$$L(Y_0,Y_0')Y_1 = \big[Y_0,\big[Y_0',Y_1\big]\big], \qquad Y_1 \in \mathrm{m}_1.$$

Then, we have

Proposition 3. *Let Y_0, $Y_0' \in \mathrm{a}+\mathrm{m}_2$. Then:*

(1) $L(Y_0,Y_0')\mathrm{m}_1 \subset \mathrm{m}_1$. *The transpose of $L(Y_0,Y_0')$ with respect to $(\ ,\ )$ is given by $L(Y_0',Y_0)$, i.e., ${}^tL(Y_0,Y_0') = L(Y_0',Y_0)$.*

(2) *Let $\mathbf{1}_1$ be the identity map of* m_1. *Then:*

(2*a*) $L(Y_0,Y_0') + L(Y_0',Y_0) = -2\big(\mu,\mu\big)\big(Y_0,Y_0'\big)\mathbf{1}_1$;

(2*b*) $L(Y_0,Y_0')\cdot L(Y_0',Y_0) = \big(\mu,\mu\big)^2\big(Y_0,Y_0\big)\big(Y_0',Y_0'\big)\mathbf{1}_1$.

Proof. The assertion (1) is clear from (2.1) and the $\operatorname{ad}\mathrm{g}$-invariance of $(\ ,\ )$. Let $Y_1 \in \mathrm{m}_1$. Since $\big[Y_0,Y_1\big] \in \mathrm{k}_1$, we have $\big[Y_0',\big[Y_0',\big[Y_0,Y_1\big]\big]\big] = -\big(\mu,\mu\big)\big(Y_0',Y_0'\big)\big[Y_0,Y_1\big]$ (see (2.4)).

Hence, by applying $\operatorname{ad} Y_0$ to this equality, we easily have (2*b*). The equality (2*a*) directly follows from (2.3). Λ

Here, we recall the notion of pseudo-abelian subspace of m. Let Q be a subspace of m. Q is called *pseudo-abelian* if it satisfies $[Q, Q] \subset \mathrm{k}_0$ (see [6]).

Proposition 4. (1) *Any subspace Q of m_2 is pseudo-abelian.*

(2) *Let Q be a pseudo-abelian subspace satisfying $Q \not\subset \mathrm{m}_2$. Then, $\dim Q \leq 2$.*

Accordingly, the inequality $\dim Q \leq 3$ holds for any pseudo-abelian subspace Q, and the equality holds when and only when $Q = \mathrm{m}_2$.

Proof. Since $[\mathrm{m}_2, \mathrm{m}_2] \subset \mathrm{k}_0$ (see (2.1)), it follows that any subspace of m_2 is pseudo-abelian. On the contrary, we already proved in Lemma 5.4 of [6] that for a pseudo-abelian subspace Q with $Q \not\subset \mathrm{m}_2$ it holds $\dim Q \leq 1 + n(\mu)$, where $n(\mu)$ means the local pseudo-nullity of the restricted root μ. (For the definition of the local pseudo-nullity, see §3 of [6].) In the case $G/K = P^2(\boldsymbol{H})$, we have $n(\mu) = 1$ (see Theorem 3.2 and Table 3 of [6]). Hence, we have $\dim Q \leq 2$. Λ

For later use, we obtain the normal form of a 2-dimensional pseudo-abelian subspace Q with $Q \not\subset \mathrm{m}_2$.

Proposition 5. *Let ξ_1 and η_1 be elements of m_1 satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$ and $(\xi_1, \eta_1) = 0$. Then, the 2-dimensional subspace $Q\,(\subset \mathrm{m})$ defined by*

$$Q = \boldsymbol{R}\,(\mu + \xi_1) + \boldsymbol{R}\left(\eta_1 + \frac{1}{4(\mu, \mu)^2}[\mu, [\xi_1, \eta_1]]\right) \tag{2.5}$$

is pseudo-abelian and $Q \not\subset \mathrm{m}_2$.

Conversely, if Q is a pseudo-abelian subspace of m with $Q \not\subset \mathrm{m}_2$ and $\dim Q = 2$, then Q can be written in the form (2.5) by utilizing suitable elements ξ_1 and $\eta_1 \in \mathrm{m}_1$ satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$ and $(\xi_1, \eta_1) = 0$.

Proof. Let ξ_1 and η_1 be elements of m_1 satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$ and $(\xi_1, \eta_1) = 0$. Then, the subspace Q defined by (2.5) satisfies $Q \not\subset \mathrm{m}_2$ and $\dim Q = 2$. Set $\eta_2 = (1/4(\mu, \mu)^2)[\mu, [\xi_1, \eta_1]]$. Then, it is easily verified that $\eta_2 \in \mathrm{m}_2$. We now show that Q is pseudo-abelian. By (2.3) and $(\xi_1, \eta_1) = 0$, we have $[\xi_1, [\eta_1, \mu]] = -[\eta_1, [\xi_1, \mu]]$. Hence, by the Jacobi identity we have

$$[\mu, [\xi_1, \eta_1]] = [[\mu, \xi_1], \eta_1] + [\xi_1, [\mu, \eta_1]] = -2[\xi_1, [\eta_1, \mu]].$$

Consequently, we have $\eta_2 = -(1/2(\mu,\mu)^2)[\xi_1,[\eta_1,\mu]]$. Note that $[\eta_1,\mu] \in \mathrm{k}_1$. Then, by the formula (2.4) and the assumption $(\xi_1,\xi_1) = 2(\mu,\mu)$ we have

$$[\xi_1,\eta_2] = -\frac{1}{2(\mu,\mu)^2}[\xi_1,[\xi_1,[\eta_1,\mu]]] = \frac{(\xi_1,\xi_1)}{2(\mu,\mu)}[\eta_1,\mu] = -[\mu,\eta_1].$$

Moreover, since $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}$ and since

$$[\mu,[\mu,\eta_2] + [\xi_1,\eta_1]] = -4(\mu,\mu)^2\eta_2 + [\mu,[\xi_1,\eta_1]] = 0,$$

it follows that $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}_0$. (Note that an element $X \in \mathrm{k}$ belongs to k_0 if and only if $[\mu,X] = 0$.) By these relations we have

$$[\mu+\xi_1,\eta_1+\eta_2] = [\mu,\eta_1] + [\xi_1,\eta_2] + [\mu,\eta_2] + [\xi_1,\eta_1] = 0 + [\mu,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}_0.$$

Since $Q = \boldsymbol{R}(\mu+\xi_1) + \boldsymbol{R}(\eta_1+\eta_2)$, this implies that Q is a pseudo-abelian subspace.

We next prove the converse. Let Q be a pseudo-abelian subspace with $Q \not\subset \mathrm{m}_2$ and $\dim Q = 2$. Then, viewing the proof of Lemma 5.4 of [6], we know that $Q \cap \mathrm{m}_2 = 0$ and $\dim(Q \cap (\mathrm{m}_1 + \mathrm{m}_2)) \leq n(\mu) = 1$. Consequently, we have $Q \not\subset \mathrm{m}_1 + \mathrm{m}_2$, because $\dim Q = 2$. Therefore, there is a basis $\{\xi,\eta\}$ of Q written in the form $\xi = \mu + \xi_1 + \xi_2$, $\eta = \eta_1 + \eta_2$, where ξ_1, $\eta_1 \in \mathrm{m}_1$, ξ_2, $\eta_2 \in \mathrm{m}_2$. Here, we note that $\eta_1 \neq 0$, because $Q \cap \mathrm{m}_2 = 0$. Subtracting a constant multiple of η from ξ if necessary, we may assume that $(\xi_1,\eta_1) = 0$. Since

$$[\xi,\eta] = [\mu+\xi_2,\eta_1] + [\xi_1,\eta_2] + [\mu+\xi_2,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}_0$$

and since $[\mu+\xi_2,\eta_1] + [\xi_1,\eta_2] \in \mathrm{k}_1$, $[\mu+\xi_2,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}_0 + \mathrm{k}_2$ and $[\xi_2,\eta_2] \in \mathrm{k}_0$, it follows that

$$[\mu+\xi_2,\eta_1] + [\xi_1,\eta_2] = 0, \tag{2.6}$$

$$[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathrm{k}_0. \tag{2.7}$$

Applying $\operatorname{ad}\mu$ to (2.7), we have $\eta_2 = (1/4(\mu,\mu)^2)[\mu,[\xi_1,\eta_1]]$. By this equality and the assumption $(\xi_1,\eta_1) = 0$, we can deduce $[\xi_1,\eta_2] = ((\xi_1,\xi_1)/2(\mu,\mu))[\eta_1,\mu]$ (see the arguments stated above). Putting this into (2.6), we have

$$[(1 - \frac{(\xi_1,\xi_1)}{2(\mu,\mu)})\mu + \xi_2,\eta_1] = 0.$$

Since $\eta_1 \neq 0$ and $\operatorname{rank}(P^2(\boldsymbol{H})) = 1$, we have $(1 - (\xi_1,\xi_1)/2(\mu,\mu))\,\mu + \xi_2 = 0$. This proves $(\xi_1,\xi_1) = 2(\mu,\mu)$ and $\xi_2 = 0$, completing the proof of the converse. Λ

3. The Gauss equation

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\boldsymbol{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $S^2\mathrm{m}^*\otimes\boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on m. We call the following equation on $\boldsymbol{\Psi}\in S^2\mathrm{m}^*\otimes\boldsymbol{N}$ the *Gauss equation* associated with $\boldsymbol{N}$:

$$([[X,Y],Z],W)=\langle\boldsymbol{\Psi}(X,Z),\boldsymbol{\Psi}(Y,W)\rangle-\langle\boldsymbol{\Psi}(X,W),\boldsymbol{\Psi}(Y,Z)\rangle, \tag{3.1}$$

where X, Y, Z, $W\in\mathrm{m}$. We denote by $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ the set of all solutions of (3.1), which is called the *Gaussian variety* associated with $\boldsymbol{N}$.

As in the case of $P^2(\boldsymbol{Cay})$ (Theorem 11 of [8]), we can prove the following

Theorem 6. *Let $\boldsymbol{N}$ be a euclidean vector space with $\dim\boldsymbol{N}=6$. Let $\boldsymbol{\Psi}\in S^2\mathrm{m}^*\otimes\boldsymbol{N}$ be a solution of the Gauss equation (3.1), i.e., $\boldsymbol{\Psi}\in\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. Then:*

(1) *There are linearly independent vectors $\mathbf{A}$ and $\mathbf{B}\in\boldsymbol{N}$ satisfying*

- (i) $\langle\mathbf{A},\mathbf{A}\rangle=\langle\mathbf{B},\mathbf{B}\rangle=4(\mu,\mu)$ *and* $\langle\mathbf{A},\mathbf{B}\rangle=2(\mu,\mu)$;
- (ii) $\boldsymbol{\Psi}(Y_0,Y_0')=(Y_0,Y_0')\mathbf{A}$, $\quad\forall Y_0,Y_0'\in\mathrm{a}+\mathrm{m}_2$;
- (iii) $\boldsymbol{\Psi}(Y_1,Y_1')=(Y_1,Y_1')\mathbf{B}$, $\quad\forall Y_1,Y_1'\in\mathrm{m}_1$;
- (iv) $\langle\mathbf{A},\boldsymbol{\Psi}(\mu,\mathrm{m}_1)\rangle=\langle\mathbf{B},\boldsymbol{\Psi}(\mu,\mathrm{m}_1)\rangle=0$.

(2) $\boldsymbol{\Psi}(Y_1,Y_2)=-\dfrac{1}{(\mu,\mu)^2}\boldsymbol{\Psi}(\mu,L(\mu,Y_2)Y_1)$, $\qquad\forall Y_1\in\mathrm{m}_1$, $\forall Y_2\in\mathrm{m}_2$.

(3) $\langle\boldsymbol{\Psi}(\mu,Y_1),\boldsymbol{\Psi}(\mu,Y_1')\rangle=(\mu,\mu)^2(Y_1,Y_1')$, $\qquad\forall Y_1,Y_1'\in\mathrm{m}_1$.

Let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2\mathrm{m}^*\otimes\boldsymbol{N}$ by

$$(h\boldsymbol{\Psi})(X,Y)=h(\boldsymbol{\Psi}(X,Y)),$$

where $\boldsymbol{\Psi}\in S^2\mathrm{m}^*\otimes\boldsymbol{N}$, $h\in O(\boldsymbol{N})$. It is easily seen that $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ is invariant under this action, i.e., $h\,\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})=\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ for any $h\in O(\boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ is *EOS* if $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})\neq\emptyset$ and if $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ is consisting of essentially one solution, i.e., for any solutions $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'\in\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$, there is an element $h\in O(\boldsymbol{N})$ satisfying $\boldsymbol{\Psi}'=h\boldsymbol{\Psi}$ (see [8]).

By Theorem 6 we can show

Theorem 7. *Let $\boldsymbol{N}$ be a euclidean vector space with $\dim\boldsymbol{N}=6$. Then, $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ is EOS.*

Proof. The proof of this theorem is quite similar to that of Theorem 10 in [8].

First we note that $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})\neq\emptyset$, because the second fundamental form of the canonical isometric imbedding $\boldsymbol{f}_0$ at the origin $o\in P^2(\boldsymbol{H})$ satisfies (3.1).

Let $\{E_i\,(1 \le i \le 4)\}$ be an orthonormal basis of m_1. (Note that $\dim \mathrm{m}_1 = 4$.) Let $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $\mathbf{A}$, $\mathbf{B}$ be the vectors of $\boldsymbol{N}$ stated in Theorem 6. We define vectors $\{\mathbf{F}_i\,(1 \le i \le 6)\}$ of $\boldsymbol{N}$ by setting $\mathbf{F}_i = \boldsymbol{\Psi}(\mu, E_i)/(\mu,\mu)\,(1 \le i \le 4)$, $\mathbf{F}_5 = (\mathbf{A}+\mathbf{B})/2\sqrt{3}|\mu|$ and $\mathbf{F}_6 = (\mathbf{A}-\mathbf{B})/2|\mu|$. By Theorem 6 we can show that $\{\mathbf{F}_i\,(1 \le i \le 6)\}$ forms an orthonormal basis of $\boldsymbol{N}$. Now let $\boldsymbol{\Psi}'$ be another element of $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be the vectors stated in Theorem 6 for $\boldsymbol{\Psi}'$. As in the case of $\boldsymbol{\Psi}$ we can also define an orthonormal basis $\{\mathbf{F}'_i\,(1 \le i \le 6)\}$ of $\boldsymbol{N}$. Then, there is an element $h \in O(6)$ satisfying $\mathbf{F}'_i = h\mathbf{F}_i\,(1 \le i \le 6)$. Here, we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\boldsymbol{\Psi}'(\mu, E_i) = h\boldsymbol{\Psi}(\mu, E_i)\,(1 \le i \le 4)$. Set $\boldsymbol{\Phi} = \boldsymbol{\Psi}' - h\boldsymbol{\Psi} \in S^2\mathrm{m}^* \otimes \boldsymbol{N}$. Then, by Theorem 6 (1) we have

$$\boldsymbol{\Phi}(\mathrm{a}+\mathrm{m}_2, \mathrm{a}+\mathrm{m}_2) = \boldsymbol{\Phi}(\mathrm{m}_1, \mathrm{m}_1) = \boldsymbol{\Phi}(\mathrm{a}, \mathrm{m}_1) = 0.$$

By Theorem 6 (2) and by the fact $L(\mu, \mathrm{m}_2)\mathrm{m}_1 \subset \mathrm{m}_1$ we have

$$\boldsymbol{\Phi}(\mathrm{m}_2, \mathrm{m}_1) \subset \boldsymbol{\Phi}(\mu, L(\mu, \mathrm{m}_2)\mathrm{m}_1) \subset \boldsymbol{\Phi}(\mathrm{a}, \mathrm{m}_1) = 0,$$

which proves $\boldsymbol{\Phi}(\mathrm{m}_2, \mathrm{m}_1) = 0$. Therefore, we have $\boldsymbol{\Phi} = 0$, i.e., $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$, completing the proof of Theorem 7. Λ

By Theorem 7 we know that $P^2(\boldsymbol{H})$ is formally rigid in codimension 6 in the sense of Agaoka–Kaneda [8]. Therefore, Theorem 1 can be obtained by Theorem 7 and the rigidity theorem (Theorem 5 of [8]).

Before proceeding to the proof of Theorem 6, we make several preparations.

Let $\boldsymbol{N}$ be a euclidean vector space. In what follows we assume $\dim \boldsymbol{N} = 6$. Let $S^2\mathrm{m}^* \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on m. Let $\boldsymbol{\Psi} \in S^2\mathrm{m}^* \otimes \boldsymbol{N}$ and $Y \in \mathrm{m}$. We define a linear map $\boldsymbol{\Psi}_Y$ of m to $\boldsymbol{N}$ by

$$\boldsymbol{\Psi}_Y\colon \mathrm{m} \ni Y' \longmapsto \boldsymbol{\Psi}(Y, Y') \in \boldsymbol{N},$$

and denote by $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ the kernel of $\boldsymbol{\Psi}_Y$. We call an element $Y \in \mathrm{m}$ *singular* (resp. *non-singular*) with respect to $\boldsymbol{\Psi}$ if $\boldsymbol{\Psi}_Y(\mathrm{m}) \ne \boldsymbol{N}$ (resp. $\boldsymbol{\Psi}_Y(\mathrm{m}) = \boldsymbol{N}$).

Let $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $Y \in \mathrm{m}\,(Y \ne 0)$. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, as shown in the proof of Proposition 5 of [7], the subspace $Q_Y = \mathrm{Ad}(k)^{-1}\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ is a pseudo-abelian subspace of m.

Proposition 8. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ *and let* $Y \in \mathrm{m}\,(Y \ne 0)$. *Then:*

(1) $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 2$ *or* 3. *Moreover,* Y *is non-singular* (*resp. singular*) *with respect to* $\boldsymbol{\Psi}$ *if and only if* $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 2$ (*resp.* $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 3$).

(2) *Let* $k \in K$ *satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Then,* $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathrm{m}_2$. *Consequently,* Y *is non-singular* (*resp. singular*) *with respect to* $\boldsymbol{\Psi}$ *if and only if* $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathrm{m}_2$ (*resp.* $\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathrm{m}_2$).

Remark 1. Recall that in the case of the Cayley projective plane $P^2(\boldsymbol{Cay})$ the inclusion $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathrm{m}_2$ in Proposition 8 (2) can be proved by a simple discussion. There, the inclusion automatically follows from the fact that any high-dimensional pseudo-abelian subspace must be contained in m_2 (see Propositions 8 and 12 of [8]). In contrast, it is not a simple task to show the inclusion $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathrm{m}_2$ in our case $P^2(\boldsymbol{H})$. We will prove this inclusion by making use of the normal form of the pseudo-abelian subspaces not contained in m_2 (see Proposition 5).

Proof of Proposition 8. Let $Y \in \mathrm{m}\,(Y \neq 0)$. Set $Q_Y = \mathrm{Ad}(k)^{-1}\mathbf{Ker}(\boldsymbol{\Psi}_Y)$, where $k \in K$ is an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Since Q_Y is pseudo-abelian, it follows that $\dim Q_Y \leq 3$ (see Proposition 4). Hence, $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \leq 3$. On the other hand, since $\dim \boldsymbol{N} = 6$ and $\dim \mathrm{m} = 8$, it follows that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq 2$. Therefore, Y is non-singular (resp. singular) with respect to $\boldsymbol{\Psi}$ if and only if $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 2$ (resp. $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 3$). This proves (1).

To show the first statement of (2) it suffices to prove $Q_Y \subset \mathrm{m}_2$. Now, let us suppose the contrary, i.e., $Q_Y \not\subset \mathrm{m}_2$. Then, we have $\dim Q_Y = 2$ (see (1) and Proposition 4 (2)). Hence, there is a basis $\{\xi, \eta\}$ of Q_Y written in the form $\xi = \mu + \xi_1$, $\eta = \eta_1 + (1/4(\mu,\mu)^2)[\mu, [\xi_1, \eta_1]]$, where ξ_1 and η_1 are elements of m_1 satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$, $(\xi_1, \eta_1) = 0$ (see Proposition 5). Let $\{\zeta_1^1, \zeta_1^2\}$ be a basis of the orthogonal complement of $\boldsymbol{R}\xi_1 + \boldsymbol{R}\eta_1$ in m_1. Set $\zeta^i = \zeta_1^i + (1/4(\mu,\mu)^2)[\mu, [\xi_1, \zeta_1^i]]$ $(i = 1, 2)$. Since $[\mu, [\xi_1, \zeta_1^i]] \in \mathrm{m}_2\,(i = 1, 2)$, we know that the vectors ζ^1 and ζ^2 are linearly independent. More strongly, they are linearly independent modulo Q_Y, i.e., $Q_Y \cap (\boldsymbol{R}\zeta^1 + \boldsymbol{R}\zeta^2) = 0$. Moreover, by Proposition 5 we know that the subspace $Q^i = \boldsymbol{R}\xi + \boldsymbol{R}\zeta^i\,(i = 1, 2)$ is also pseudo-abelian, because $(\xi_1, \zeta_1^i) = 0$. Consequently, we have $[[\xi, \zeta^i], \mu] = 0\,(i = 1, 2)$.

Set $X = \mathrm{Ad}(k)\xi$, $Z^i = \mathrm{Ad}(k)\zeta^i\,(i = 1, 2)$. Then, we have $X \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)\,(X \neq 0)$, $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$ and $[[X, Z^i], Y] = 0\,(i = 1, 2)$. By the Gauss equation (3.1) we have

$$0 = ([[X, Z^i], Y], W) = \langle \boldsymbol{\Psi}(X, Y), \boldsymbol{\Psi}(Z^i, W) \rangle - \langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Z^i, Y) \rangle, \quad (i = 1, 2),$$

where W is an arbitrary element of m. Since $\boldsymbol{\Psi}_Y(X) = 0$, we obtain by this equality $\langle \boldsymbol{\Psi}_X(W), \boldsymbol{\Psi}(Z^i, Y) \rangle = 0$, i.e., $\langle \boldsymbol{\Psi}_X(\mathrm{m}), \boldsymbol{\Psi}(Z^i, Y) \rangle = 0\,(i = 1, 2)$. We note that the vectors $\boldsymbol{\Psi}(Z^1, Y)$ and $\boldsymbol{\Psi}(Z^2, Y)$ are linearly independent, because $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$. Hence, we have $\dim \boldsymbol{\Psi}_X(\mathrm{m}) \leq \dim \boldsymbol{N} - 2 = 4$, implying $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq 4$. This contradicts the assertion (1). Thus, we have $Q_Y \subset \mathrm{m}_2$, proving the first statement of (2). The last statement of (2) is now clear. Λ

As a corollary of Proposition 8 we obtain

Proposition 9. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Then:*

(1) *Let* $Y_0 \in \mathrm{a} + \mathrm{m}_2\,(Y_0 \neq 0)$. *Then,* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{\xi \in \mathrm{a} + \mathrm{m}_2 \,|\, (\xi, Y_0) = 0\}$. *If* Y_0 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \{\xi \in \mathrm{a} + \mathrm{m}_2 \,|\, (\xi, Y_0) = 0\}$.

(2) *Let* $Y_1 \in \mathrm{m}_1\,(Y_1 \neq 0)$. *Then,* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) \subset \{\eta \in \mathrm{m}_1 \,|\, (\eta, Y_1) = 0\}$. *If* Y_1 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) = \{\eta \in \mathrm{m}_1 \,|\, (\eta, Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathrm{a} + \mathrm{m}_2\,(Y_0 \neq 0)$. Then, we can take an element $k_0 \in K$ such that $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathrm{m}_2) = \{\xi \in \mathrm{a} + \mathrm{m}_2 \,|\, (\xi, Y_0) = 0\}$ (see Proposition 7 of [7]). This proves (1). Similarly, for $Y_1 \in \mathrm{m}_1\,(Y_1 \neq 0)$, we can easily show (2). Λ

Let $\boldsymbol{\Psi} \in S^2\mathrm{m}^* \otimes \boldsymbol{N}$. We call a subspace U of m *singular* with respect to $\boldsymbol{\Psi}$ if each element of U is singular with respect to $\boldsymbol{\Psi}$.

Proposition 10. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Assume that* $Y \in \mathrm{m}\,(Y \neq 0)$ *is non-singular with respect to* $\boldsymbol{\Psi}$. *Then, there is a non-zero vector* $\mathbf{E} \in \boldsymbol{N}$ *such that*

$$\boldsymbol{N} = \boldsymbol{R}\mathbf{E} + \boldsymbol{\Psi}_\xi(\mathrm{m}) \ (\textit{orthogonal direct sum}) \tag{3.2}$$

holds for any $\xi \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)\,(\xi \neq 0)$. *Consequently,* $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ *is a singular subspace with respect to* $\boldsymbol{\Psi}$.

Proof. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, since Y is non-singular, we have $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathrm{m}_2$. Take a non-zero element satisfying $Y' \in \mathrm{Ad}(k)\mathrm{m}_2$ and $Y' \notin \mathbf{Ker}(\boldsymbol{\Psi}_Y)$ and set $\mathbf{E} = \boldsymbol{\Psi}(Y, Y')\,(\neq 0)$. Let $\xi \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)\,(\xi \neq 0)$. Then, by the Gauss equation (3.1) we have

$$\big([[\xi, Y'], Y], W\big) = \big\langle \boldsymbol{\Psi}(\xi, Y), \boldsymbol{\Psi}(Y', W)\big\rangle - \big\langle \boldsymbol{\Psi}(\xi, W), \boldsymbol{\Psi}(Y', Y)\big\rangle,$$

where W is an arbitrary element of m. Here, we note that $[[\xi, Y'], Y] = 0$, because $[[\xi, Y'], Y] \in \mathrm{Ad}(k)[[\mathrm{m}_2, \mathrm{m}_2], \mu] = 0$. Since $\boldsymbol{\Psi}(\xi, Y) = 0$, we obtain by the above equality $\langle \mathbf{E}, \boldsymbol{\Psi}(\xi, W)\rangle = 0$. This shows $\langle \mathbf{E}, \boldsymbol{\Psi}_\xi(\mathrm{m})\rangle = 0$ and hence $\boldsymbol{\Psi}_\xi(\mathrm{m}) \neq \boldsymbol{N}$. Consequently, ξ is singular with respect to $\boldsymbol{\Psi}$. Since $\dim \mathbf{Ker}(\boldsymbol{\Psi}_\xi) = 3$ (see Proposition 8), we have $\dim \boldsymbol{\Psi}_\xi(\mathrm{m}) = 5$, which proves the decomposition (3.2). Λ

4. Proof of Theorem 6

In this section, with the preparations in the previous sections, we will prove Theorem 6. We first show

Proposition 11. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Then, there are singular subspaces* $U\,(\subset \mathrm{a} + \mathrm{m}_2)$ *and* $V\,(\subset \mathrm{m}_1)$ *with respect to* $\boldsymbol{\Psi}$ *satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$.

Proof. If $\mathrm{a}+\mathrm{m}_2$ contains no non-singular element with respect to $\boldsymbol{\Psi}$, then set $U=\mathrm{a}+\mathrm{m}_2$. On the contrary, if there is a non-singular element $Y_0 \in \mathrm{a}+\mathrm{m}_2$, then set $U=\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0})$. In this case we know that $\dim U=2$, $U \subset \mathrm{a}+\mathrm{m}_2$ and that U is a singular subspace with respect to $\boldsymbol{\Psi}$ (see Proposition 8, Proposition 9 and Proposition 10).

Similarly, we can show that there is a singular subspace V of m_1 with respect to $\boldsymbol{\Psi}$ satisfying the desired properties. Λ

Proposition 12. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. *Let* $U\,(\subset \mathrm{a}+\mathrm{m}_2)$ *and* $V\,(\subset \mathrm{m}_1)$ *be singular subspaces with respect to* $\boldsymbol{\Psi}$ *satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$. *Then, there are vectors* $\mathbf{A},\ \mathbf{B} \in \boldsymbol{N}$ *such that*:

(1) $\big\langle \mathbf{A},\mathbf{A}\big\rangle = \big\langle \mathbf{B},\mathbf{B}\big\rangle = 4\big(\mu,\mu\big)$.

(2) *Let* $\xi \in U$ *and* $\eta \in V$. *Then*:

(2*a*) $\boldsymbol{\Psi}(\xi,Y_0) = \big(\xi,Y_0\big)\mathbf{A}, \quad \forall Y_0 \in \mathrm{a}+\mathrm{m}_2;$

(2*b*) $\boldsymbol{\Psi}(\eta,Y_1) = \big(\eta,Y_1\big)\mathbf{B}, \quad \forall Y_1 \in \mathrm{m}_1.$

(3) *Let* $Y_0 \in \mathrm{a}+\mathrm{m}_2$ *and* $Y_1 \in \mathrm{m}_1$. *Then*:

(3*a*) $\big\langle \mathbf{A},\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1)\big\rangle = \big\langle \mathbf{B},\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1)\big\rangle = 0;$

(3*b*) $\big\langle \mathbf{A},\boldsymbol{\Psi}_{Y_1}(\mathrm{a}+\mathrm{m}_2)\big\rangle = \big\langle \mathbf{B},\boldsymbol{\Psi}_{Y_1}(\mathrm{a}+\mathrm{m}_2)\big\rangle = 0.$

(4) *Let* $\xi \in U\ (\xi \neq 0)$ *and* $\eta \in V\ (\eta \neq 0)$. *Then*:

(4*a*) $\boldsymbol{\Psi}_{\xi}(\mathrm{m}) = \boldsymbol{R}\mathbf{A} + \boldsymbol{\Psi}_{\xi}(\mathrm{m}_1)$ (*orthogonal direct sum*);

(4*b*) $\boldsymbol{\Psi}_{\eta}(\mathrm{m}) = \boldsymbol{R}\mathbf{B} + \boldsymbol{\Psi}_{\eta}(\mathrm{a}+\mathrm{m}_2)$ (*orthogonal direct sum*).

(5) *Let* $Y_0 \in \mathrm{a}+\mathrm{m}_2$ *and* $Y_1 \in \mathrm{m}_1$. *Then*:

(5*a*) $\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\mathbf{A}\big\rangle = 4\big(\mu,\mu\big)\big(Y_0,Y_0\big);$

(5*b*) $\big\langle \boldsymbol{\Psi}(Y_1,Y_1),\mathbf{B}\big\rangle = 4\big(\mu,\mu\big)\big(Y_1,Y_1\big).$

(6) *Let* $\xi \in U$, $\eta \in V$, $Y_0 \in \mathrm{a}+\mathrm{m}_2$ *and* $Y_1 \in \mathrm{m}_1$. *Assume that* $\big(\xi,Y_0\big) = \big(\eta,Y_1\big) = 0$. *Then*:

(6*a*) $\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}_{\xi}(\mathrm{m}_1)\big\rangle = 0;$

(6*b*) $\big\langle \boldsymbol{\Psi}(Y_1,Y_1),\boldsymbol{\Psi}_{\eta}(\mathrm{a}+\mathrm{m}_2)\big\rangle = 0.$

Proof. The assertions (1), (2) and (3) can be proved in the same manner as in the proof of Proposition 16 of [8]. Hence, we omit their proofs.

Let $\xi \in U\ (\xi \neq 0)$. By (2*a*) we easily get $\boldsymbol{\Psi}_{\xi}(\mathrm{a}+\mathrm{m}_2) = \boldsymbol{R}\mathbf{A}$ and hence $\boldsymbol{\Psi}_{\xi}(\mathrm{m}) = \boldsymbol{R}\mathbf{A} + \boldsymbol{\Psi}_{\xi}(\mathrm{m}_1)$. Since $\big\langle \mathbf{A},\boldsymbol{\Psi}_{\xi}(\mathrm{m}_1)\big\rangle = 0$ (see (3*a*)), we have the decomposition (4*a*). Similarly, we can show (4*b*).

The assertions (5*a*) and (6*a*) are proved as follows: Let $Y_0 \in \mathrm{a}+\mathrm{m}_2$. Take $\xi \in U\ (\xi \neq 0)$ such that $\big(\xi,Y_0\big) = 0$. Then, we have $\big[\big[Y_0,\xi\big],Y_0\big] = 4\big(\mu,\mu\big)\big(Y_0,Y_0\big)\xi$ (see (2.2)) and $\boldsymbol{\Psi}(\xi,Y_0) = 0$ (see (2*a*)). By the Gauss equation (3.1) we have

$$\big(\big[\big[Y_0,\xi\big],Y_0\big],\xi\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}(\xi,\xi)\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,\xi),\boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$

$$\big([[Y_0,\xi],Y_0],Y_1'\big)=\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}(\xi,Y_1')\big\rangle-\big\langle \boldsymbol{\Psi}(Y_0,Y_1'),\boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$

where Y_1' is an arbitrary element of m_1. By these equalities we have $\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\mathbf{A}\big\rangle = 4\big(\mu,\mu\big)\big(Y_0,Y_0\big)$ and $\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}(\xi,Y_1')\big\rangle=0$. Therefore, we obtain (5a) and (6a). The assertions (5b) and (6b) can be proved in a similar way. Λ

Remark 2. As seen in the proof of Proposition 11, singular subspaces U and V may not be uniquely determined. However, the vectors $\mathbf{A}$ and $\mathbf{B}$ in Proposition 8 do not depend on the choice of singular subspaces U and V, which will be clarified at the last part of this section (see Lemma 20).

In the following argument, we take and fix an element $\boldsymbol{\Psi}\in\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. We denote by U and V singular subspaces with respect to $\boldsymbol{\Psi}$ satisfying $U\,(\subset \mathrm{a}+\mathrm{m}_2)$, $V\,(\subset \mathrm{m}_1)$, $\dim U\geq 2$ and $\dim V\geq 2$. We also denote by $\mathbf{A}$, $\mathbf{B}$ the vectors of $\boldsymbol{N}$ obtained by applying Proposition 12 to the pair of singular subspaces U and V.

Lemma 13. (1) *Let $Y_0\in \mathrm{a}+\mathrm{m}_2$. Then:*

$$\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1),\boldsymbol{\Psi}_{Y_0}(Y_1')\big\rangle=\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}(Y_1,Y_1')\big\rangle-\big(\mu,\mu\big)\big(Y_0,Y_0\big)\big(Y_1,Y_1'\big),\quad \forall Y_1,\,Y_1'\in\mathrm{m}_1.$$

(2) *Let $Y_0\in \mathrm{a}+\mathrm{m}_2$ and $\xi\in U$ satisfy $\big(\xi,Y_0\big)=0$. Then:*

$$\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1),\boldsymbol{\Psi}_{\xi}(Y_1')\big\rangle=\big(L(Y_0,\xi)Y_1,Y_1'\big),\quad \forall Y_1,\,Y_1'\in\mathrm{m}_1.$$

Proof. Putting $X=Y_0$, $Y=Y_1$, $Z=Y_0$, $W=Y_1'$ into (3.1), we have

$$\big([[Y_0,Y_1],Y_0],Y_1'\big)=\big\langle \boldsymbol{\Psi}(Y_0,Y_0),\boldsymbol{\Psi}(Y_1,Y_1')\big\rangle-\big\langle \boldsymbol{\Psi}(Y_0,Y_1'),\boldsymbol{\Psi}(Y_1,Y_0)\big\rangle.$$

Since $\big[Y_0,\big[Y_0,Y_1\big]\big]=-\big(\mu,\mu\big)\big(Y_0,Y_0\big)Y_1$ (see (2.2)), we easily get (1).

Similarly, putting $X=\xi$, $Y=Y_1$, $Z=Y_0$ and $W=Y_1'$ into (3.1), we have

$$\begin{aligned}\big([[\xi,Y_1],Y_0],Y_1'\big)&=\big\langle \boldsymbol{\Psi}(\xi,Y_0),\boldsymbol{\Psi}(Y_1,Y_1')\big\rangle-\big\langle \boldsymbol{\Psi}(\xi,Y_1'),\boldsymbol{\Psi}(Y_1,Y_0)\big\rangle\\&=\big\langle \mathbf{A},\boldsymbol{\Psi}(Y_1,Y_1')\big\rangle\big(\xi,Y_0\big)-\big\langle \boldsymbol{\Psi}_{\xi}(Y_1'),\boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle.\end{aligned}$$

Since $\big(\xi,Y_0\big)=0$, we have

$$\big\langle \boldsymbol{\Psi}_{\xi}(Y_1'),\boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle=-\big([[\xi,Y_1],Y_0],Y_1'\big)=\big(L(Y_0,\xi)Y_1,Y_1'\big),$$

proving (2). Λ

Let $\xi\in U$ $(\xi\neq 0)$. Since $\dim\mathbf{Ker}(\boldsymbol{\Psi}_{\xi})=3$ (see Proposition 8) and since $\dim\mathrm{m}=8$, we have $\dim\boldsymbol{\Psi}_{\xi}(\mathrm{m})=5$. Let us denote by $\boldsymbol{E}_{\xi}$ the one dimensional orthogonal complement of $\boldsymbol{\Psi}_{\xi}(\mathrm{m})$ in $\boldsymbol{N}$.

Proposition 14. *Set* $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$. *Then:*

(1) *Let* $\xi \in U$. *Then:*

$$\langle \boldsymbol{\Psi}_\xi(Y_1), \boldsymbol{\Psi}_\xi(\eta) \rangle = C(\xi, \xi)(Y_1, \eta), \qquad \forall Y_1 \in \mathrm{m}_1,\ \forall \eta \in V. \tag{4.1}$$

(2) *The inequality* $0 < C \leq 3(\mu, \mu)$ *holds. The vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *are linearly independent if* $C \neq 3(\mu, \mu)$ *and* $\mathbf{A} = \mathbf{B}$ *if* $C = 3(\mu, \mu)$.

(3) *Let* $\xi \in U$ $(\xi \neq 0)$. *Then,* $\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathrm{m}_1)$, $\quad \forall Y_0 \in \mathrm{a} + \mathrm{m}_2$.

(4) *If* $C \neq 3(\mu, \mu)$, *then:*

$$\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1) = \boldsymbol{\Psi}_\xi(\mathrm{m}_1), \qquad \forall Y_0 \in \mathrm{a} + \mathrm{m}_2\ (Y_0 \neq 0), \forall \xi \in U\ (\xi \neq 0); \tag{4.2}$$

$$\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_0 \in \mathrm{a} + \mathrm{m}_2; \tag{4.3}$$

$$\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_1 \in \mathrm{m}_1. \tag{4.4}$$

Proof. Put $Y_0 = \xi$ and $Y_1' = \eta$ into Lemma 13 (1). Then, since $\boldsymbol{\Psi}(\xi, \xi) = (\xi, \xi)\mathbf{A}$ and $\boldsymbol{\Psi}(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we get (4.1).

In view of Proposition 12 (1), we easily have $\langle \mathbf{A}, \mathbf{B} \rangle \leq 4(\mu, \mu)$ and hence $C \leq 3(\mu, \mu)$. Further, by putting $Y_1 = \eta\,(\neq 0)$ into (4.1) we know $C > 0$, because $\boldsymbol{\Psi}_\xi(\eta) \neq 0$ (see Proposition 9). This shows $\langle \mathbf{A}, \mathbf{B} \rangle > (\mu, \mu)$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if $\langle \mathbf{A}, \mathbf{B} \rangle \neq 4(\mu, \mu)$, i.e., $C \neq 3(\mu, \mu)$. It is easy to see that if $C = 3(\mu, \mu)$, i.e., $\langle \mathbf{A}, \mathbf{B} \rangle = 4(\mu, \mu)$, then $\mathbf{A} = \mathbf{B}$.

We next prove (3). Let $\xi \in U$ $(\xi \neq 0)$. By Proposition 12 ($4a$) we know that the orthogonal complement of $\boldsymbol{R}\mathbf{A}$ in $\boldsymbol{N}$ is given by $\boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathrm{m}_1)$. Hence, by Proposition 12 ($3a$), we have $\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathrm{m}_1)$ for any $Y_0 \in \mathrm{a} + \mathrm{m}_2$.

Finally, we prove (4). Since $C \neq 3(\mu, \mu)$, the subspace $\boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ forms a 2-dimensional subspace of $\boldsymbol{N}$. Let $Y_0 \in \mathrm{a} + \mathrm{m}_2\ (Y_0 \neq 0)$. Then, by Proposition 12 ($3a$) we know that $\boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1)$ coincides with the orthogonal complement of $\boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ in $\boldsymbol{N}$. (Recall that $\dim \boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1) = 4$ and $\dim \boldsymbol{N} = 6$.) Let $\xi \in U$ $(\xi \neq 0)$. Since $\boldsymbol{\Psi}_\xi(\mathrm{m}_1)$ is also an orthogonal complement of $\boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$, it follows that $\boldsymbol{\Psi}_\xi(\mathrm{m}_1) = \boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1)$. If we take $\xi \in U$ $(\xi \neq 0)$ satisfying $(\xi, Y_0) = 0$, then by Proposition 12 ($6a$) we obtain $\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$. Similarly, we can prove $\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ for any $Y_1 \in \mathrm{m}_1$, completing the proof of (4). Λ

Let $Y_0 \in \mathrm{a} + \mathrm{m}_2$ and $\xi \in U$ $(\xi \neq 0)$. Define a linear mapping $\boldsymbol{\Theta}_{Y_0, \xi} \colon \mathrm{m}_1 \longrightarrow \boldsymbol{N}$ by

$$\boldsymbol{\Theta}_{Y_0, \xi}(Y_1) = \boldsymbol{\Psi}_{Y_0}(Y_1) + \frac{1}{C(\xi, \xi)} \boldsymbol{\Psi}_\xi(L(\xi, Y_0)Y_1), \quad Y_1 \in \mathrm{m}_1. \tag{4.5}$$

Then, we have

Proposition 15. *Let* $Y_0 \in \mathrm{a}+\mathrm{m}_2$, $\xi \in U$ $(\xi \neq 0)$ *and* $Y_1 \in \mathrm{m}_1$. *Assume that* $(\xi, Y_0) = 0$ *and* $L(\xi, Y_0)Y_1 \in V$. *Then:*

(1) $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. *More strongly, if* $C \neq 3(\mu,\mu)$, *then* $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.

(2) $|\boldsymbol{\Theta}_{Y_0,\xi}(Y_1)|^2 = \langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1)\rangle - (\mu,\mu)\left\{1+(\mu,\mu)/C\right\}(Y_0,Y_0)(Y_1,Y_1)$.

Proof. By Proposition 14 (3) we know that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathrm{m}_1)$. Here, we note that $\langle \boldsymbol{E}_\xi, \boldsymbol{\Psi}_\xi(\mathrm{m}_1)\rangle = 0$, because $\boldsymbol{E}_\xi$ is orthogonal to $\boldsymbol{\Psi}_\xi(\mathrm{m})$. Let $Y_1' \in \mathrm{m}_1$. Then, by Lemma 13 (2), Proposition 14 (1) and Proposition 3 (2) we have

$$\begin{aligned}\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\rangle &= \langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\rangle + \frac{1}{C(\xi,\xi)}\langle \boldsymbol{\Psi}_\xi(L(\xi,Y_0)Y_1), \boldsymbol{\Psi}_\xi(Y_1')\rangle \\ &= (L(Y_0,\xi)Y_1, Y_1') + (L(\xi,Y_0)Y_1, Y_1') \\ &= 0,\end{aligned}$$

proving $\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathrm{m}_1)\rangle = 0$. This implies that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. In the case where $C \neq 3(\mu,\mu)$, we have $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{\Psi}_{Y_0}(\mathrm{m}_1) + \boldsymbol{\Psi}_\xi(\mathrm{m}_1) = \boldsymbol{\Psi}_\xi(\mathrm{m}_1)$ (see (4.2)), which proves $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.

Next, we show (2). By Lemma 13 and by the equality $\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathrm{m}_1)\rangle = 0$, we have

$$\begin{aligned}\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Theta}_{Y_0,\xi}(Y_1)\rangle &= \langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle \\ &= \langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle + \frac{1}{C(\xi,\xi)}\langle \boldsymbol{\Psi}_\xi(L(\xi,Y_0)Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle \\ &= \langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1)\rangle - (\mu,\mu)(Y_0,Y_0)(Y_1,Y_1) \\ &\qquad + \frac{1}{C(\xi,\xi)}(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y_1).\end{aligned}$$

On the other hand, by Proposition 3 we have

$$\begin{aligned}(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y_1) &= (L(\xi,Y_0)L(\xi,Y_0)Y_1, Y_1) \\ &= -(L(Y_0,\xi)L(\xi,Y_0)Y_1, Y_1) \\ &= -(\mu,\mu)^2(\xi,\xi)(Y_0,Y_0)(Y_1,Y_1).\end{aligned}$$

Therefore, we get the assertion (2). Λ

With these preparations we begin with the proof Theorem 6. First, we consider the case $\dim V = 2$.

Lemma 16. *Assume that* $\dim V = 2$. *Then,* $C \neq 3(\mu,\mu)$. *Accordingly, the vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *are linearly independent.*

Proof. Take non-zero elements ξ, $\xi' \in U$ satisfying $(\xi,\xi')=0$. Then, by Proposition 3 (2) it follows that $L(\xi,\xi')=-L(\xi',\xi)$ and $L(\xi,\xi')$ gives an isomorphism of m_1 onto itself. Let $Y_1\in L(\xi,\xi')V$. Then, by Proposition 3 (2*b*) we have $L(\xi,\xi')Y_1\in V$. Hence, by Proposition 15 (1) we have $\mathbf{\Theta}_{\xi',\xi}(Y_1)\in \boldsymbol{E}_\xi$. Since $\dim L(\xi,\xi')V=\dim V=2$ and $\dim \boldsymbol{E}_\xi=1$, it is possible to take a non-zero element $Y_1\in L(\xi,\xi')V$ satisfying $\mathbf{\Theta}_{\xi',\xi}(Y_1)=0$. Therefore, by Proposition 15 (2) and Proposition 12 (2*a*) we have

$$0=|\mathbf{\Theta}_{\xi',\xi}(Y_1)|^2=\left[\left\langle \mathbf{\Psi}(Y_1,Y_1),\mathbf{A}\right\rangle-\left(\mu,\mu\right)\left\{1+\left(\mu,\mu\right)/C\right\}\left(Y_1,Y_1\right)\right]\left(\xi',\xi'\right).$$

Since $(\xi',\xi')\neq 0$, we have

$$\left\langle \mathbf{\Psi}(Y_1,Y_1),\mathbf{A}\right\rangle=\left(\mu,\mu\right)\left\{1+\left(\mu,\mu\right)/C\right\}\left(Y_1,Y_1\right). \tag{4.6}$$

Now, we suppose the case $C=3(\mu,\mu)$. Then, by (4.6) we have $\left\langle \mathbf{\Psi}(Y_1,Y_1),\mathbf{A}\right\rangle=\frac{4}{3}\left(\mu,\mu\right)\left(Y_1,Y_1\right)$. On the other hand, by Proposition 12 (5*b*) we have $\left\langle \mathbf{\Psi}(Y_1,Y_1),\mathbf{A}\right\rangle=4\left(\mu,\mu\right)\left(Y_1,Y_1\right)$, because $\mathbf{A}=\mathbf{B}$ in case $C=3(\mu,\mu)$ (see Proposition 14 (2)). Hence, we have $(Y_1,Y_1)=0$, which contradicts the assumption $Y_1\neq 0$. Therefore, we have $C\neq 3(\mu,\mu)$ and hence $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. Λ

Lemma 17. *Assume that* $\dim V=2$. *Then,* V *can be extended to a* 3*-dimensional singular subspace contained in* m_1*, i.e., there is a singular subspace* $\widehat{V}$ $(\subset \mathrm{m}_1)$ *such that* $V\subset\widehat{V}$ *and* $\dim\widehat{V}=3$.

Proof. Let $\mathbf{F}\in \boldsymbol{R}\mathbf{A}+\boldsymbol{R}\mathbf{B}$ be a unit vector which is orthogonal to $\mathbf{B}$. Then, for any $\eta\in V$ we have $\left\langle \mathbf{F},\mathbf{\Psi}_\eta(\mathrm{m})\right\rangle=0$, because $\left\langle \mathbf{F},\mathbf{\Psi}_\eta(\mathrm{m})\right\rangle=\left\langle \mathbf{F},\boldsymbol{R}\mathbf{B}+\mathbf{\Psi}_\eta(\mathrm{a}+\mathrm{m}_2)\right\rangle=0$ (see Proposition 12 (4*b*) and (3*b*)).

Now, define a symmetric bilinear form χ on m_1 by setting

$$\chi(Y_1,Y_1')=\left\langle \mathbf{\Psi}(Y_1,Y_1'),\mathbf{F}\right\rangle,\qquad Y_1,Y_1'\in\mathrm{m}_1.$$

Since $\mathbf{\Psi}(Y_1,Y_1')\in\boldsymbol{R}\mathbf{B}+\boldsymbol{R}\mathbf{F}$ (see Proposition 14 (4)) and $\left\langle \mathbf{\Psi}(Y_1,Y_1'),\mathbf{B}\right\rangle=\left\langle \mathbf{B},\mathbf{B}\right\rangle\left(Y_1,Y_1'\right)$ for Y_1, $Y_1'\in\mathrm{m}_1$ (see Proposition 12 (5)), we have

$$\mathbf{\Psi}(Y_1,Y_1')=\left(Y_1,Y_1'\right)\mathbf{B}+\chi(Y_1,Y_1')\mathbf{F},\qquad Y_1,Y_1'\in\mathrm{m}_1. \tag{4.7}$$

Let $V^\perp$ be the orthogonal complement of V in m_1. Then, we have $\dim V^\perp=2$. (Recall that $\dim\mathrm{m}_1=4$ and $\dim V=2$.) Let $\{Y_1,Y_1'\}$ be an orthonormal basis of $V^\perp$. Then, putting $X=Z=Y_1$ and $Y=W=Y_1'$ into the Gauss equation (3.1), we have

$$\begin{aligned}\left(\left[\left[Y_1,Y_1'\right],Y_1\right],Y_1'\right)&=\left\langle \mathbf{B},\mathbf{B}\right\rangle\left(Y_1,Y_1\right)\left(Y_1',Y_1'\right)\\&\qquad+\chi(Y_1,Y_1)\chi(Y_1',Y_1')-\chi(Y_1,Y_1')\chi(Y_1',Y_1).\end{aligned}$$

Since $\left(\left[\left[Y_1,Y_1'\right],Y_1\right],Y_1'\right)=\left\langle \mathbf{B},\mathbf{B}\right\rangle\left(Y_1,Y_1\right)\left(Y_1',Y_1'\right)$ (see (2.2)), we have

$$\chi(Y_1,Y_1)\chi(Y_1',Y_1')-\chi(Y_1,Y_1')\chi(Y_1',Y_1)=0.$$

This implies that χ is degenerate on $V^\perp$. Therefore, there is a non-zero vector $\zeta \in V^\perp$ such that $\chi(\zeta, V^\perp) = 0$, i.e., $\langle \mathbf{F}, \boldsymbol{\Psi}_\zeta(V^\perp)\rangle = 0$.

Let us show that the subspace $\widehat{V} = \boldsymbol{R}\zeta + V\ (\subset \mathrm{m}_1)$ is singular with respect to $\boldsymbol{\Psi}$. Note that $\langle \mathbf{F}, \boldsymbol{\Psi}_\zeta(\mathrm{a} + \mathrm{m}_2)\rangle = 0$ (see Proposition 12 (3*b*)). Then, since $\mathrm{m} = \mathrm{a} + \mathrm{m}_2 + V + V^\perp$ and $\boldsymbol{\Psi}_\zeta(V) \subset \boldsymbol{R}\mathbf{B}$, it follows that

$$\langle \mathbf{F}, \boldsymbol{\Psi}_\zeta(\mathrm{m})\rangle = \langle \mathbf{F}, \boldsymbol{\Psi}_\zeta(\mathrm{a}+\mathrm{m}_2) + \boldsymbol{\Psi}_\zeta(V) + \boldsymbol{\Psi}_\zeta(V^\perp)\rangle \subset 0 + \langle \mathbf{F}, \boldsymbol{R}\mathbf{B}\rangle + 0 = 0.$$

Hence, we have $\langle \mathbf{F}, \boldsymbol{\Psi}_{a\zeta+\eta}(\mathrm{m})\rangle = 0$ for any $a \in \boldsymbol{R}$ and $\eta \in V$. Consequently, $\boldsymbol{\Psi}_{a\zeta+\eta}(\mathrm{m}) \neq \boldsymbol{N}$, which implies that $a\zeta + \eta \in \widehat{V}$ is singular with respect to $\boldsymbol{\Psi}$. Λ

Now, we assume that $\dim V = 2$ and denote by $\widehat{V}$ be the singular subspace stated in the above lemma. Let $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ be the vectors obtained by applying Proposition 12 to the pair of singular subspaces U and $\widehat{V}$. Then, by Proposition 12 (2) we can easily see that $\widehat{\mathbf{A}} = \mathbf{A}$ and $\widehat{\mathbf{B}} = \mathbf{B}$. Therefore, we know that all the statements in Proposition 12 and hence the arguments developed after Proposition 12 are also true if we simply replace V by $\widehat{V}$. Accordingly, without loss of generality we can assume that $\dim V \geq 3$.

Lemma 18. $\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B}\rangle = (\mu,\mu)\left\{1 + (\mu,\mu)/C\right\}(Y_0, Y_0), \quad \forall Y_0 \in \mathrm{a} + \mathrm{m}_2.$

Proof. As in the proof of Lemma 16, we can prove that $C \neq 3(\mu,\mu)$. Let $Y_0 \in \mathrm{a}+\mathrm{m}_2$ ($Y_0 \neq 0$). Take $\xi \in U$ ($\xi \neq 0$) such that $(\xi, Y_0) = 0$, which is possible because $\dim U \geq 2$. Then, by Proposition 3 (2) it follows that $L(\xi, Y_0) = -L(Y_0, \xi)$ and that the map $L(\xi, Y_0)$ gives an isomorphism of m_1 onto itself. Now, take $\eta \in V$ ($\eta \neq 0$) such that $L(\xi, Y_0)\eta \in V$. This is also possible because $\dim L(\xi, Y_0)V = \dim V \geq 3$ and $\dim(V \cap L(\xi, Y_0)V) \geq 2$. (Note that $\dim \mathrm{m}_1 = 4$.) Then, by Proposition 15 and Proposition 12 (2*b*) we have

$$0 = |\boldsymbol{\Theta}_{Y_0,\xi}(\eta)|^2 = \left[\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B}\rangle - (\mu,\mu)\left\{1 + (\mu,\mu)/C\right\}(Y_0, Y_0)\right](\eta,\eta).$$

Since $(\eta,\eta) \neq 0$, we get the lemma. Λ

Lemma 19. *$C = (\mu,\mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$.*

Proof. Take $\xi \in U$ ($\xi \neq 0$). Then, by Lemma 18 and $\boldsymbol{\Psi}(\xi,\xi) = (\xi,\xi)\mathbf{A}$ (see Proposition 12 (2*a*)), we have $\langle \mathbf{A}, \mathbf{B}\rangle = (\mu,\mu)\left\{1 + (\mu,\mu)/C\right\}$. Since $C = \langle \mathbf{A}, \mathbf{B}\rangle - (\mu,\mu)$, we easily have $C^2 = (\mu,\mu)^2$. Moreover, since $C > 0$ (see Proposition 14 (2)), it follows that $C = (\mu,\mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$. Λ

Now, we show

Lemma 20. (1) $\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0, Y_0' \in \mathrm{a} + \mathrm{m}_2.$

(2) $\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1, Y_1' \in \mathrm{m}_1.$

Proof. On account of an elementary fact concerning symmetric bilinear forms, we have only to show $\mathbf{\Psi}(Y_0, Y_0) = \big(Y_0, Y_0\big)\mathbf{A}$ and $\mathbf{\Psi}(Y_1, Y_1) = \big(Y_1, Y_1\big)\mathbf{B}$ for any $Y_0 \in \mathrm{a} + \mathrm{m}_2$ and $Y_1 \in \mathrm{m}_1$.

Let $Y_0 \in \mathrm{a} + \mathrm{m}_2$. Then, by Lemma 18 and Lemma 19 we have $\big\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{B} \big\rangle = \big\langle \mathbf{A}, \mathbf{B} \big\rangle \big(Y_0, Y_0\big)$. Moreover, by Proposition 12 (1) and (5a) we have $\big\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{A} \big\rangle = \big\langle \mathbf{A}, \mathbf{A} \big\rangle \big(Y_0, Y_0\big)$. Since $\mathbf{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ (see (4.3)), it follows that $\mathbf{\Psi}(Y_0, Y_0) = \big(Y_0, Y_0\big)\mathbf{A}$, which proves (1).

We next prove (2). Let $Y_1 \in \mathrm{m}_1$ ($Y_1 \neq 0$). Take elements $\xi \in U$ ($\xi \neq 0$) and $\eta \in V$ ($\eta \neq 0$) such that $\big(\eta, Y_1\big) = 0$. Set $Y_0 = \big[Y_1, [\xi, \eta]\big]$. Then, it is easy to see that $\big[\xi, \eta\big] \in \mathrm{k}_1$ and $Y_0 \in \mathrm{a} + \mathrm{m}_2$ (see (2.1)). Further, we have $\big(\xi, Y_0\big) = 0$ and $L(\xi, Y_0)Y_1 \in V$, because

$$
\begin{aligned}
\big(\xi, Y_0\big) &= \big(\xi, \big[Y_1, [\xi, \eta]\big]\big) = -\big(\big[\xi, [\xi, \eta]\big], Y_1\big) = \big(\mu, \mu\big)\big(\xi, \xi\big)\big(\eta, Y_1\big) = 0, \\
L(\xi, Y_0)Y_1 &= \big[\xi, \big[\big[Y_1, [\xi, \eta]\big], Y_1\big]\big] = \big(\mu, \mu\big)\big(Y_1, Y_1\big)\big[\xi, [\xi, \eta]\big] \\
&= -\big(\mu, \mu\big)^2\big(\xi, \xi\big)\big(Y_1, Y_1\big)\eta \in V
\end{aligned}
$$

(see (2.2) and (2.4)). Thus, by Proposition 15 (2), Lemma 19 and $\mathbf{\Psi}(Y_0, Y_0) = \big(Y_0, Y_0\big)\mathbf{A}$ (see (1)), we have

$$
0 = |\mathbf{\Theta}_{Y_0, \xi}(Y_1)|^2 = \big[\big\langle \mathbf{A}, \mathbf{\Psi}(Y_1, Y_1) \big\rangle - 2\big(\mu, \mu\big)\big(Y_1, Y_1\big)\big]\ \big(Y_0, Y_0\big).
$$

Here, we note that $Y_0 \neq 0$, because $L(\xi, Y_0)Y_1 \neq 0$. Hence, by the above equality and Lemma 19, we have $\big\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{A} \big\rangle = \big\langle \mathbf{B}, \mathbf{A} \big\rangle \big(Y_1, Y_1\big)$. On the other hand, by Proposition 12 (1) and (5b) we have $\big\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{B} \big\rangle = \big\langle \mathbf{B}, \mathbf{B} \big\rangle \big(Y_1, Y_1\big)$. Consequently, it follows that $\mathbf{\Psi}(Y_1, Y_1) = \big(Y_1, Y_1\big)\mathbf{B}$, because $\mathbf{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ (see (4.4)). This proves (2). Λ

We are now in a final position of the proof of Theorem 6. Let $Y_0 \in \mathrm{a} + \mathrm{m}_2$ ($Y_0 \neq 0$). Then, by Lemma 20 (1) we have $\mathbf{Ker}(\mathbf{\Psi}_{Y_0}) \supset \{Y_0' \in \mathrm{a} + \mathrm{m}_2 \mid \big(Y_0, Y_0'\big) = 0\}$. This shows $\dim \mathbf{Ker}(\mathbf{\Psi}_{Y_0}) \geq 3$ and hence Y_0 is singular with respect to $\mathbf{\Psi}$ (see Proposition 9 (1)). Accordingly, $\mathrm{a} + \mathrm{m}_2$ is a singular subspace. Similarly, by Lemma 20 (2) we can show that m_1 is also a singular subspace.

Now, let us put into Proposition 12 $U = \mathrm{a} + \mathrm{m}_2$ and $V = \mathrm{m}_1$. Then, by Lemma 20 we know that the vectors $\mathbf{A}$ and $\mathbf{B}$ are not altered by this change of singular subspaces. Therefore, all the statements in Proposition 12 and the arguments developed after Proposition 12 are also true under our setting $U = \mathrm{a} + \mathrm{m}_2$ and $V = \mathrm{m}_1$. Consequently, by Proposition 12 (1), (2), (3) and Lemma 19 we get the assertion (1) of Theorem 6. We also obtain by Proposition 14 and $C = \big(\mu, \mu\big)$ (see Lemma 19) the assertion (3) of Theorem 6.

Finally, we prove the assertion (2) of Theorem 6. Let $Y_2 \in \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then, since $C \neq 3(\mu, \mu)$ and $(\mu, Y_2) = 0$, we have

$$\boldsymbol{\Theta}_{Y_2,\mu}(Y_1) = \boldsymbol{\Psi}_{Y_2}(Y_1) + \frac{1}{(\mu,\mu)^2}\boldsymbol{\Psi}_{\mu}(L(\mu, Y_2)Y_1) = 0$$

(see Proposition 15). Here we note that the conditions $\mu \in U$ and $L(\mu, Y_2)Y_1 \in V$ in Proposition 15 have no significance, because $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Accordingly, we obtain the assertion (2). This completes the proof of Theorem 6. Λ

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