Generating functions of Littlewood-Richardson coefficients

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Abstract

We give generating functions of the Littlewood-Richardson coefficients expressing the product of two Schur functions for the cases $\{\lambda_1, \cdots, \lambda_m\}\{\mu_1\}$ $(m \geq 1)$ and $\{\lambda_1, \cdots, \lambda_m\}\{\mu_1, \mu_2\}$ $(m \leq 4)$. These formulas also give generating functions expressing the number of irreducible components of $\{\lambda\}\{\mu\}$ for these cases. As an application, we give a new decomposition formula of the product $\{\lambda_1, \lambda_2\}\{\mu_1, \mu_2\}$, expressed as a linear combination of some basic partitions whose coefficients move in some polytopes. Several conjectures on the generating function for the case $\{\lambda_1, \cdots, \lambda_m\}\{\mu_1, \mu_2\}$ $(m \geq 5)$ and a simple formula expressing the Littlewood-Richardson polynomial for the case $\{\lambda\} = \{\lambda_1, \lambda_2\}$, $\{\mu\} = \{\mu_1, \mu_2\}$ are also stated.

1. Introduction.

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the coefficient of S_ν in the product of two Schur function S_λ and S_μ, i.e., $S_\lambda S_\mu = \sum c^{\nu}_{\lambda\mu} S_\nu$. (Hereafter, we use the classical notation $\{\lambda\}\{\mu\}$ instead of $S_\lambda S_\mu$). This coefficient is also equal to the multiplicity of V_ν in the tensor product $V_\lambda \otimes V_\mu$, where V_λ etc. means the irreducible representation space of the general linear group $GL(V)$ corresponding to the partition $\{\lambda\}$ etc. The coefficient $c^{\nu}_{\lambda\mu}$ also appears in many fields of mathematics such as combinatorics, representation theory, algebraic geometry, etc. The value $c^{\nu}_{\lambda\mu}$ can be calculated by the Littlewood-Richardson rule, which requires a combinatorial procedure on Young diagrams. Other several methods to calculate $c^{\nu}_{\lambda\mu}$ are also known. For example, it is known that the coefficient $c^{\nu}_{\lambda\mu}$ can be expressed as a number of lattice points in some convex polytopes (cf. [5], [6], [12]). But

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in general it is almost impossible to express the individual value $c^{\nu}_{\lambda\mu}$ as a function of $\{\lambda\}$, $\{\mu\}$ and $\{\nu\}$, because it requires many case by case examinations. (Exceptionally, some special formulas are known. See [19], [22], [23], [24] etc.)

The purpose of this paper is to give several new formulas expressing the explicit values of $c^{\nu}_{\lambda\mu}$. Our main tool is a "generating function", which seems to be the most natural language to describe the values $c^{\nu}_{\lambda\mu}$. One generating function contains all informations on $c^{\nu}_{\lambda\mu}$'s when the depths of $\{\lambda\}$ and $\{\mu\}$ are fixed. This phenomenon is surprising because the Littlewood-Richardson rule itself is a combinatorial algorithm on Young diagrams, and because this result means that the totality of such combinatorial calculations is condensed into one generating function.

There are already several attempts to calculate generating functions of $c^{\nu}_{\lambda\mu}$ such as [4], [17], etc. But it seems that the main interests of these papers are devoted to find an efficient algorithm to calculate them, not directed to their explicit forms nor their properties. But after some calculations, we are convinced that generating functions possess several peculiar properties deserve to more attention. The main purpose of this paper is to give their explicit expressions and to explain their properties, mainly on two types of products $\{\lambda_1,\cdots,\lambda_m\}\,\{\mu_1\}$ $(m\geq 1)$ and $\{\lambda_1,\cdots,\lambda_m\}\{\mu_1,\mu_2\}$ $(m\leq 4)$ (Theorems 1 and 4). We also state several conjectures for the second type product for $m\geq 5$.

In addition, as one application of such generating functions, we give a new proof of the result of Stembridge [26], characterizing the pair (λ,μ) such that the product $\{\lambda\}\{\mu\}$ is multiplicity-free in case the depths of $\{\lambda\}$ and $\{\mu\}\leq 2$ (Theorem 8). For these $\{\lambda\}$ and $\{\mu\}$ we also give a quite simple formula expressing the Littlewood-Richardson polynomial (Theorem 9).

2. The case $\{\lambda_1,\cdots,\lambda_m\}\{\mu_1\}$.

In this section, we first consider the generating function for the product $\{\lambda\}\{\mu\} = \{\lambda_1,\cdots,\lambda_m\}\{\mu_1\}$. In this case each component of the product $\{\lambda\}\{\mu\}$ consists of partitions of the form

$$\{\nu\}=\{|\lambda|+\mu_1-|\nu|,\nu_1,\cdots,\nu_m\},$$

where $|\lambda|=\lambda_1+\cdots+\lambda_m$ and $|\nu|=\nu_1+\cdots+\nu_m$. The decomposition can be calculated by Pieri's formula, which is a special case of the Littlewood-Richardson rule [25]. We express the multiplicity of $\{\nu\}$ in $\{\lambda\}\{\mu\}$ as $c^{\nu}_{\lambda\mu}$ and consider the following generating function expressing the decomposition of $\{\lambda\}\{\mu\}$:

$$F_{m,1}=\sum_{\lambda,\mu,\nu} c^{\nu}_{\lambda\mu}\, {x_1}^{\nu_1}\cdots {x_m}^{\nu_m} {q_1}^{\lambda_1}\cdots {q_m}^{\lambda_m} {r_1}^{\mu_1}.$$

Namely the coefficient of ${x_1}^{\nu_1}\cdots {x_m}^{\nu_m} {q_1}^{\lambda_1}\cdots {q_m}^{\lambda_m} {r_1}^{\mu_1}$ of this function is equal to the multiplicity of $\{\nu\}$ in $\{\lambda\}\{\mu\}$. (In the case $\{\mu\}=\{\mu_1\}$, the product $\{\lambda\}\{\mu\}$ is multiplicity-free and this coefficient is actually 0 or 1.) For fixed m, this single function contains all informations on the decomposition of the product $\{\lambda\}\{\mu\}$ for $\{\lambda\}$ with depth$\leq m$ and $\{\mu\}$ with depth≤ 1. We consider the problem expressing this generating function $F_{m,1}$ in a simple form. The answer is given in the following theorem.

Theorem 1. *We have*

$$F_{m,1} = \frac{1}{(1-q_1)(1-x_1q_1q_2)(1-x_1x_2q_1q_2q_3)\cdots(1-x_1\cdots x_{m-1}q_1\cdots q_m) \times (1-r_1)(1-x_1q_1r_1)(1-x_1x_2q_1q_2r_1)\cdots(1-x_1\cdots x_mq_1\cdots q_mr_1)}.$$

Note that if we put $x_1 = \cdots = x_m = 1$ into these functions, we obtain the generating functions expressing the number of irreducible components of $\{\lambda\}\{\mu\}$.

Corollary 2. *The number of irreducible components of* $\{\lambda_1, \cdots, \lambda_m\}\{\mu_1\}$ *is equal to the coefficient of* ${q_1}^{\lambda_1}\cdots{q_m}^{\lambda_m}{r_1}^{\mu_1}$ *in the following function*:

$$\frac{1}{(1-q_1)(1-q_1q_2)(1-q_1q_2q_3)\cdots(1-q_1\cdots q_m) \times (1-r_1)(1-q_1r_1)(1-q_1q_2r_1)\cdots(1-q_1\cdots q_mr_1)}.$$

Example. We consider the case $m = 3$. In this case the function $F_{3,1}$ is expanded as:

$$\begin{aligned} F_{3,1} = {} & 1 + q_1 + r_1 + {q_1}^2 + x_1q_1q_2 + (1+x_1)q_1r_1 + {r_1}^2 + {q_1}^3 + x_1{q_1}^2q_2 + x_1x_2q_1q_2q_3 \\ & + (1+x_1){q_1}^2r_1 + (x_1 + x_1x_2)q_1q_2r_1 + (1+x_1)q_1{r_1}^2 + {r_1}^3 + \cdots\cdots \\ & + (x_1x_2 + x_1x_2x_3 + {x_1}^2x_2 + {x_1}^2x_2x_3){q_1}^2q_2q_3{r_1}^2 + \cdots\cdots . \end{aligned}$$

From the coefficients of q_1r_1 and ${q_1}^2q_2q_3{r_1}^2$, we obtain the following decompositions:

$$\begin{aligned} &\{1\}\{1\} = \{2\} + \{1^2\}, \\ &\{21^2\}\{2\} = \{41^2\} + \{31^3\} + \{321\} + \{2^21^2\}. \end{aligned}$$

If we put $x_1 = x_2 = x_3 = 1$ into $F_{3,1}$, we obtain

$$\begin{aligned} & 1 + q_1 + r_1 + {q_1}^2 + q_1q_2 + 2q_1r_1 + {r_1}^2 + {q_1}^3 + {q_1}^2q_2 + q_1q_2q_3 + 2{q_1}^2r_1 + 2q_1q_2r_1 + 2q_1{r_1}^2 \\ & \quad + {r_1}^3 + \cdots + 8{q_1}^3{q_2}^2q_3{r_1}^4 + \cdots\cdots . \end{aligned}$$

Hence, for example, we know that the number of irreducible components of $\{321\}\{4\}$ is 8.

We give another decomposition formula of $\{\lambda_1, \cdots, \lambda_m\}\{\mu_1\}$ as a corollary of Theorem 1, which is similar to the decomposition formulas of plethysms stated in [1], [2].

Corollary 3. *We have the following decomposition formula*:

$$\{\lambda_1, \lambda_2, \cdots, \lambda_m\}\{\mu_1\} = \sum_{\begin{cases} a_1+b_1=\lambda_1-\lambda_2 \\ a_2+b_2=\lambda_2-\lambda_3 \\ \cdots\cdots\cdots \\ a_{m-1}+b_{m-1}=\lambda_{m-1}-\lambda_m \\ a_m+b_m=\lambda_m \\ a_0+a_1+\cdots+a_m=\mu_1 \\ a_i, b_j \geq 0 \end{cases}} \Big[(a_0+b_1)\{1,0,\cdots,0\} + (a_1+b_2)\{1,1,0,\cdots,0\} + \cdots\cdots + (a_{m-1}+b_m)\{1,1,\cdots,1,0\} + a_m\{1,1,\cdots,1\} \Big].$$

Example. We consider the case $\{421\}\{2\}$. In this case the set of non-negative integers satisfying the conditions

$$\begin{cases} a_1+b_1 = 4-2 = 2, \\ a_2+b_2 = 2-1 = 1, \\ a_3+b_3 = 1, \\ a_0+a_1+a_2+a_3 = 2 \end{cases}$$

is given as follows:

a_0	a_1	a_2	a_3	b_1	b_2	b_3	$\{\nu\}$
2	0	0	0	2	1	1	$\{621\}$
1	0	0	1	2	1	0	$\{521^2\}$
1	0	1	0	2	0	1	$\{52^2\}$
0	0	1	1	2	0	0	$\{42^21\}$
1	1	0	0	1	1	1	$\{531\}$
0	1	0	1	1	1	0	$\{431^2\}$
0	1	1	0	1	0	1	$\{432\}$
0	2	0	0	0	1	1	$\{4^21\}$

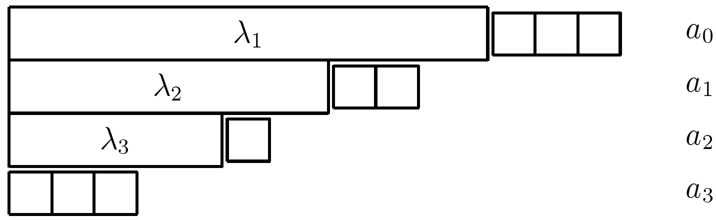
And the right column gives the desired decomposition of $\{421\}\{2\}$.

Now we give a proof of Theorem 1 and Corollary 3.

Proof of Theorem 1. We fix $\{\lambda\} = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and $\{\mu\} = \{\mu_1\}$. To each row of the Young diagram $\{\lambda\}$ we add a_0, $a_1, \cdots, a_m$ new boxes with $a_0 + \cdots + a_m = \mu_1$ respectively as follows (this figure corresponds to the case $m = 3$).

From the Littlewood-Richardson rule the integers a_i must satisfy the following conditions:

$$\begin{cases} a_1 \leq \lambda_1 - \lambda_2, \\ a_2 \leq \lambda_2 - \lambda_3, \\ \quad \cdots\cdots \\ a_{m-1} \leq \lambda_{m-1} - \lambda_m, \\ a_m \leq \lambda_m. \end{cases} \tag{2.1}$$



Then we have

$$F_{m,1} = \sum_{\lambda,\mu,a_i} {x_1}^{\lambda_2+a_1} {x_2}^{\lambda_3+a_2} \cdots {x_{m-1}}^{\lambda_m+a_{m-1}} {x_m}^{a_m} {q_1}^{\lambda_1} {q_2}^{\lambda_2} \cdots {q_m}^{\lambda_m} {r_1}^{\mu_1},$$

where a_i move in the above range (2.1) and satisfy $a_0 + \cdots + a_m = \mu_1$. We here define non-negative integers b_i by

$$\begin{cases} a_1 + b_1 = \lambda_1 - \lambda_2, \\ a_2 + b_2 = \lambda_2 - \lambda_3, \\ \quad \cdots\cdots \\ a_{m-1} + b_{m-1} = \lambda_{m-1} - \lambda_m, \\ a_m + b_m = \lambda_m. \end{cases}$$

Then we have

$$\begin{cases} \lambda_1 = a_1 + \cdots + a_m + b_1 + \cdots + b_m, \\ \lambda_2 = a_2 + \cdots + a_m + b_2 + \cdots + b_m, \\ \quad \cdots\cdots \\ \lambda_{m-1} = a_{m-1} + a_m + b_{m-1} + b_m, \\ \lambda_m = a_m + b_m. \end{cases}$$

Hence we have

$$\begin{aligned} F_{m,1} = \sum_{a_i,b_i} & {x_1}^{a_1+\cdots+a_m+b_2+\cdots+b_m} {x_2}^{a_2+\cdots+a_m+b_3+\cdots+b_m} \cdots {x_{m-1}}^{a_{m-1}+a_m+b_m} {x_m}^{a_m} \\ & \times {q_1}^{a_1+\cdots+a_m+b_1+\cdots+b_m} {q_2}^{a_2+\cdots+a_m+b_2+\cdots+b_m} \cdots {q_m}^{a_m+b_m} {r_1}^{a_0+\cdots+a_m} \\ = \sum_{a_i,b_i} & {q_1}^{b_1} (x_1q_1q_2)^{b_2} (x_1x_2q_1q_2q_3)^{b_3} \cdots (x_1 \cdots x_{m-2} q_1 \cdots q_{m-1})^{b_{m-1}} \\ & \times (x_1 \cdots x_{m-1} q_1 \cdots q_m)^{b_m} {r_1}^{a_0} (x_1q_1r_1)^{a_1} (x_1x_2q_1q_2r_1)^{a_2} \cdots \\ & \times (x_1 \cdots x_{m-1} q_1 \cdots q_{m-1} r_1)^{a_{m-1}} (x_1 \cdots x_m q_1 \cdots q_m r_1)^{a_m} \\ = & \frac{1}{\begin{array}{l}(1-q_1)(1-x_1q_1q_2)(1-x_1x_2q_1q_2q_3)\cdots(1-x_1\cdots x_{m-1}q_1\cdots q_m) \\ \qquad \times (1-r_1)(1-x_1q_1r_1)(1-x_1x_2q_1q_2r_1)\cdots(1-x_1\cdots x_mq_1\cdots q_mr_1).\end{array}} \end{aligned}$$

q.e.d.

Proof of Corollary 3. We consider the equality in the above proof:

$$F_{m,1}=\sum x_1{}^{a_1+\cdots+a_m+b_2+\cdots+b_m}x_2{}^{a_2+\cdots+a_m+b_3+\cdots+b_m}\cdots x_{m-1}{}^{a_{m-1}+a_m+b_m}x_m{}^{a_m}$$
$$\times\, q_1{}^{a_1+\cdots+a_m+b_1+\cdots+b_m}q_2{}^{a_2+\cdots+a_m+b_2+\cdots+b_m}\cdots q_m{}^{a_m+b_m}r_1{}^{a_0+a_1\cdots+a_m}.$$

This term corresponds to the partition

$$\{*,a_1+\cdots+a_m+b_2+\cdots+b_m,a_2+\cdots+a_m+b_3+\cdots+b_m,\cdots\cdots,a_{m-1}+a_m+b_m,a_m\}$$

in the product $\{\lambda_1,\ldots,\lambda_m\}\{\mu_1\}$, where

$$\begin{cases}\lambda_1=a_1+\cdots+a_m+b_1+\cdots+b_m,\\ \lambda_2=a_2+\cdots+a_m+b_2+\cdots+b_m,\\ \quad\cdots\cdots\cdots\\ \lambda_m=a_m+b_m,\\ \mu_1=a_0+a_1+\cdots+a_m.\end{cases}$$

It is easy to see that the value of $*$ in the above term is equal to $a_0+a_1+\cdots+a_m+b_1+\cdots+b_m$,and hence Corollary 3 follows immediately from these expressions. q.e.d.

3. The case $\{\lambda_1,\cdots,\lambda_m\}\{\mu_1,\mu_2\}$.

Next, in this section we consider the case $\{\lambda\}\{\mu\}=\{\lambda_1,\cdots,\lambda_m\}\{\mu_1,\mu_2\}$. In this case the tensor product $\{\lambda\}\{\mu\}$ consists of terms of the form $\{\nu\}=\{|\lambda|+|\mu|-|\nu|,\nu_1,\cdots,\nu_{m+1}\}$, where $|\lambda|=\lambda_1+\cdots+\lambda_m$, $|\mu|=\mu_1+\mu_2$ and $|\nu|=\nu_1+\cdots+\nu_{m+1}$. As in §2, we express the multiplicity of $\{\nu\}$ in $\{\lambda\}\{\mu\}$ as $c^{\nu}_{\lambda\mu}$ and consider the generating function

$$F_{m,2}=\sum_{\lambda,\mu,\nu}c^{\nu}_{\lambda\mu}\,x_1{}^{\nu_1}\cdots x_{m+1}{}^{\nu_{m+1}}q_1{}^{\lambda_1}\cdots q_m{}^{\lambda_m}r_1{}^{\mu_1}r_2{}^{\mu_2}.$$

Namely the coefficient of $x_1{}^{\nu_1}\cdots x_{m+1}{}^{\nu_{m+1}}q_1{}^{\lambda_1}\cdots q_m{}^{\lambda_m}r_1{}^{\mu_1}r_2{}^{\mu_2}$ is equal to the multiplicity of $\{\nu\}$ in $\{\lambda\}\{\mu\}$, which is not in general multiplicity-free in this case. We consider the problem expressing this generating function as a ratio of two polynomials. For small m (≤ 4) we can calculate $F_{m,2}$ by a similar method as in the proof of Theorem 1 (see Theorem 4 below). But for general m (≥ 5) it is still undetermined, and in the next section we state several conjectures on the form of $F_{m,2}$.

The numerator of $F_{m,2}$ is not so simple as that of $F_{m,1}$, and to express it, we first introduce one notation. Let a_1, a_2, $\cdots$, a_m be integers satisfying $a_1\geq a_2\geq\cdots\geq a_m\geq 0$. We define a polynomial $f_{a_1\cdots a_m}$ by

$$f_{a_1\cdots a_m}=\sum_{b_1,\cdots,b_m}(x_1q_1)^{b_1}\cdots(x_mq_m)^{b_m}$$

where the exponents $b_1 \geq \cdots \geq b_m \geq 0$ move in the range such that the following two sets coincide:

$$\{b_1 - b_2, b_2 - b_3, \cdots, b_{m-1} - b_m, b_m\} = \{a_1, a_2, \cdots, a_{m-1}, a_m\}.$$

For example, we have

$$\begin{aligned}
f_{211} &= (x_1q_1)^4(x_2q_2)^2(x_3q_3) + (x_1q_1)^4(x_2q_2)^3(x_3q_3) + (x_1q_1)^4(x_2q_2)^3(x_3q_3)^2,\\
f_{2110} &= (x_1q_1)^4(x_2q_2)^2(x_3q_3) + (x_1q_1)^4(x_2q_2)^2(x_3q_3)(x_4q_4) + (x_1q_1)^4(x_2q_2)^2(x_3q_3)^2(x_4q_4)\\
&\quad + (x_1q_1)^4(x_2q_2)^4(x_3q_3)^2(x_4q_4) + (x_1q_1)^4(x_2q_2)^3(x_3q_3)(x_4q_4) + (x_1q_1)^4(x_2q_2)^3(x_3q_3)\\
&\quad + (x_1q_1)^4(x_2q_2)^4(x_3q_3)^3(x_4q_4) + (x_1q_1)^4(x_2q_2)^3(x_3q_3)^3(x_4q_4) + (x_1q_1)^4(x_2q_2)^3(x_3q_3)^2\\
&\quad + (x_1q_1)^4(x_2q_2)^4(x_3q_3)^3(x_4q_4)^2 + (x_1q_1)^4(x_2q_2)^3(x_3q_3)^3(x_4q_4)^2\\
&\quad + (x_1q_1)^4(x_2q_2)^3(x_3q_3)^2(x_4q_4)^2.
\end{aligned}$$

Note that the exponent of x_1q_1 is always constant, which is equal to the sum $b_1 = \sum a_i$.

Theorem 4. *For $m = 1 \sim 4$, the generating function $F_{m,2}$ can be expressed as a ratio of two polynomials*

$$\frac{f(x_1q_1, \cdots, x_mq_m, r_1, r_2)}{g(x_1, \cdots, x_{m+1}, q_1, \cdots, q_m, r_1, r_2)}.$$

Here the numerator f is expressed as follows in terms of $f_{a_1 \cdots a_m}$:

$$\begin{aligned}
m = 1 &: f = 1,\\
m = 2 &: f = 1 - f_{11}{r_1}^2r_2,\\
m = 3 &: f = 1 - (f_{110} + f_{111}){r_1}^2r_2 + f_{111}{r_1}^3r_2 - f_{111}{r_1}^2{r_2}^2 + (f_{111} + f_{211}){r_1}^3{r_2}^2\\
&\qquad - f_{222}{r_1}^5{r_2}^3,\\
m = 4 &: f = 1 - (f_{1100} + f_{1110}){r_1}^2r_2 + (f_{1110} + f_{1111}){r_1}^3r_2 - f_{1111}{r_1}^4r_2 - (f_{1110}\\
&\qquad + f_{1111}){r_1}^2{r_2}^2 + (f_{1110} + f_{2110} + 5f_{1111} + f_{2111}){r_1}^3{r_2}^2 - (f_{1111} + f_{2111}){r_1}^4{r_2}^2\\
&\qquad + (f_{1111} + f_{2111}){r_1}^3{r_2}^3 - (f_{1111} + f_{2111} + f_{3111} + f_{2211}){r_1}^4{r_2}^3 - (f_{2220} + f_{2221}\\
&\qquad + f_{2222} + f_{2211}){r_1}^5{r_2}^3 + (f_{2221} + f_{2222}){r_1}^6{r_2}^3 - (f_{2221} + f_{2222}){r_1}^5{r_2}^4 + (f_{2221}\\
&\qquad + f_{3221} + 5f_{2222} + f_{3222}){r_1}^6{r_2}^4 - (f_{2222} + f_{3222}){r_1}^7{r_2}^4 - f_{2222}{r_1}^5{r_2}^5 + (f_{2222}\\
&\qquad + f_{3222}){r_1}^6{r_2}^5 - (f_{3222} + f_{3322}){r_1}^7{r_2}^5 + f_{3333}{r_1}^9{r_2}^6,
\end{aligned}$$

and the denominator g is given by

$$\begin{aligned}
g(x_1, \cdots, x_{m+1}, q_1, \cdots, q_m, r_1, r_2) = &\prod_{\left\{\substack{0 \leq i \leq m+1\\ 0 \leq j \leq m\\ 0 \leq k \leq 2\\ j+k = i+1}\right.} (1 - x_1 \cdots x_i q_1 \cdots q_j r_1 \cdots r_k)\\
&\times \prod_{2 \leq i+2 \leq j \leq m} \{1 - (x_1q_1)^2 \cdots (x_iq_i)^2(x_{i+1}q_{i+1}) \cdots (x_jq_j)r_1r_2\}.
\end{aligned}$$

Example. In the case $m = 2$, the above generating function is equal to

$$F_{2,2} = \frac{1 - {x_1}^2 x_2 {q_1}^2 q_2 {r_1}^2 r_2}{\begin{array}{c}(1-q_1)(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1r_1r_2)(1-x_1x_2q_1q_2r_1) \\ \times (1-x_1x_2q_1r_1r_2)(1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2)\end{array}}$$
$$= 1 + q_1 + r_1 + {q_1}^2 + x_1q_1q_2 + (x_1+1)q_1r_1 + \cdots\cdots$$
$$+ ({x_1}^3x_2 + {x_1}^3 + {x_1}^2{x_2}^2 + {x_1}^2x_2x_3 + 2{x_1}^2x_2 + {x_1}^2 + x_1x_2x_3 + x_1x_2){q_1}^3q_2{r_1}^2r_2$$
$$+ \cdots\cdots .$$

The coefficient of ${q_1}^3q_2{r_1}^2r_2$ implies that $\{31\}\{21\}$ decomposes into

$$\{3^21\} + \{43\} + \{32^2\} + \{321^2\} + 2\{421\} + \{52\} + \{41^3\} + \{51^2\}.$$

In the case $m = 3$, we put $x_1 = \cdots = x_4 = 1$. Then we have

$$f = 1 - ({q_1}^2q_2 + {q_1}^2q_2q_3 + {q_1}^2{q_2}^2q_3 + {q_1}^3{q_2}^2q_3){r_1}^2r_2 + {q_1}^3{q_2}^2q_3{r_1}^3r_2 - {q_1}^3{q_2}^2q_3{r_1}^2{r_2}^2$$
$$+ ({q_1}^3{q_2}^2q_3 + {q_1}^4{q_2}^2q_3 + {q_1}^4{q_2}^3q_3 + {q_1}^4{q_2}^3{q_3}^2){r_1}^3{r_2}^2 - {q_1}^6{q_2}^4{q_3}^2{r_1}^5{r_2}^3,$$
$$g = (1-q_1)(1-r_1)(1-q_1q_2)(1-q_1r_1)(1-r_1r_2)(1-q_1q_2q_3)(1-q_1q_2r_1)$$
$$\times (1-q_1r_1r_2)(1-q_1q_2q_3r_1)(1-q_1q_2r_1r_2)^2(1-q_1q_2q_3r_1r_2)^2(1-{q_1}^2q_2q_3r_1r_2).$$

And by using computers, we know that the coefficient of ${q_1}^6{q_2}^4{q_3}^2{r_1}^4{r_2}^2$ in f/g is 123, which is just equal to the number of irreducible components of $\{642\}\{42\}$.

In the case $m = 4$, the numerator f stated in Theorem 4 is a little lengthy, and it actually consists of 144 monomials if we express it directly as a polynomial of $x_1, \cdots, x_4$, $q_1, \cdots, q_4$, r_1, r_2.

Before proceeding to the proof of Theorem 4, we review some relations to the previously known results. Similar generating functions are already stated in [17] and [4]. In [17; p.178] Patera and Sharp gave a formula on the generating function for $SU(2)$ Clebsch-Gordan series:

$$\frac{1}{(1-A_1A)(1-A_2A)(1-A_1A_2)}.$$

This is essentially equal to the function $F_{1,1}$ given in Theorem 1. In fact we have only to change the variables by the rule $q_1 = A_1A$, $r_1 = A_2A$ and $x_1 = 1/A^2$. (See also [4; p.7612].) Similarly, the generating function for $SU(3)$ Clebsch-Gordan series stated in [17; p.178]

$$\frac{1}{(1-A_1A)(1-B_1B)(1-A_2A)(1-B_2B)(1-A_1B_2)(1-B_1A_2)}$$
$$\times \left(\frac{1}{1-A_1A_2B} + \frac{B_1B_2A}{1-B_1B_2A} \right)$$

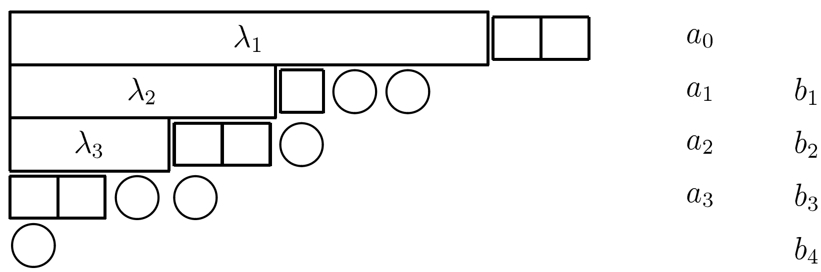
is a special case of our generating function $F_{2,2}$. (See also [4; p.7634].) In this case, we substitute

$$q_1 = A_1 A, \quad q_2 = \frac{AB_1}{A_1}, \quad r_1 = A_2 A, \quad r_2 = \frac{AB_2}{A_2}, \quad x_1 = \frac{B}{A^2}, \quad x_2 = \frac{1}{AB}, \quad x_3 = 0$$

into $F_{2,2}$. Then it just coincides with the above. (Actually, since the formula in [17] treats the group $SU(3)$, it gives the multiplicity of $\{|\lambda| + |\mu| - |\nu|, \nu_1, \nu_2, 0\}$ in $\{\lambda_1, \lambda_2\}$ $\{\mu_1, \mu_2\}$. Our formula on the function $F_{2,2}$ may be considered as a generalization of their result in [17] to more general $\nu_3 \neq 0$ case.)

Proof of Theorem 4. We here give a proof for the case $m = 3$. Other cases can be treated in the same way, and we leave its examination to the readers.

We consider the Young diagram corresponding to the partition $\{\lambda_1, \lambda_2, \lambda_3\}$. First, we add a_0, a_1, a_2, a_3 ($a_0 + \cdots + a_3 = \mu_1$) new boxes to each row, and next add b_1, b_2, b_3, b_4 ($b_1 + \cdots + b_4 = \mu_2$) boxes to the second $\sim$ the fifth row under the rule of Littlewood-Richardson as follows (b_i boxes are represented by circles in the figure):



The Littlewood-Richardson rule can be expressed as the following conditions:

$$(3.1) \qquad \begin{cases} a_0 + a_1 + a_2 + a_3 = \mu_1, \\ a_1 \leq \lambda_1 - \lambda_2, \\ a_2 \leq \lambda_2 - \lambda_3, \\ a_3 \leq \lambda_3, \\ b_1 + b_2 + b_3 + b_4 = \mu_2, \\ a_1 + b_1 + \lambda_2 \leq \lambda_1 + a_0, \\ a_2 + b_2 + \lambda_3 \leq \lambda_2 + a_1, \\ a_3 + b_3 \leq \lambda_3 + a_2, \\ b_4 \leq a_3, \\ b_1 \leq a_0, \\ b_1 + b_2 \leq a_0 + a_1, \\ b_1 + b_2 + b_3 \leq a_0 + a_1 + a_2, \\ b_1 + b_2 + b_3 + b_4 \leq a_0 + a_1 + a_2 + a_3. \end{cases}$$

Here we put

$$\begin{cases} \lambda_1 - \lambda_2 = a_1 + c_1, \\ \lambda_2 - \lambda_3 = a_2 + c_2, \\ \lambda_3 = a_3 + c_3, \\ a_3 = b_4 + c_4, \\ a_0 = b_1 + c_5. \end{cases}$$

Then from the above conditions (3.1) we have clearly $c_i \geq 0$. After some calculations we know that the conditions (3.1) can be summarized to the following (3.2) and (3.3):

$$\begin{cases} \lambda_1 = a_1 + a_2 + b_4 + c_1 + c_2 + c_3 + c_4, \\ \lambda_2 = a_2 + b_4 + c_2 + c_3 + c_4, \\ \lambda_3 = b_4 + c_3 + c_4, \\ \mu_1 = a_1 + a_2 + b_1 + b_4 + c_4 + c_5, \\ \mu_2 = b_1 + b_2 + b_3 + b_4, \\ a_3 = b_4 + c_4, \\ a_0 = b_1 + c_5. \end{cases} \tag{3.2}$$

$$\begin{cases} b_2 \leq a_1 + c_2, \\ b_3 \leq a_2 + c_3, \\ b_2 \leq a_1 + c_5, \\ b_2 + b_3 \leq a_1 + a_2 + c_5. \end{cases} \tag{3.3}$$

Then the generating function $F_{3,2}$ can be expressed as follows:

$$\begin{aligned} F_{3,2} &= \sum x_1{}^{\lambda_2+a_1+b_1} x_2{}^{\lambda_3+a_2+b_2} x_3{}^{a_3+b_3} x_4{}^{b_4} q_1{}^{\lambda_1} q_2{}^{\lambda_2} q_3{}^{\lambda_3} r_1{}^{\mu_1} r_2{}^{\mu_2} \\ &= \sum x_1{}^{a_1+a_2+b_1+b_4+c_2+c_3+c_4} x_2{}^{a_2+b_2+b_4+c_3+c_4} x_3{}^{b_3+b_4+c_4} x_4{}^{b_4} q_1{}^{a_1+a_2+b_4+c_1+c_2+c_3+c_4} \\ &\quad \times q_2{}^{a_2+b_4+c_2+c_3+c_4} q_3{}^{b_4+c_3+c_4} r_1{}^{a_1+a_2+b_1+b_4+c_4+c_5} r_2{}^{b_1+b_2+b_3+b_4} \\ &= \sum (x_1q_1r_1)^{a_1} (x_1x_2q_1q_2r_1)^{a_2} (x_1r_1r_2)^{b_1} (x_2r_2)^{b_2} (x_3r_2)^{b_3} (x_1x_2x_3x_4q_1q_2q_3r_1r_2)^{b_4} \\ &\quad \times q_1{}^{c_1} (x_1q_1q_2)^{c_2} (x_1x_2q_1q_2q_3)^{c_3} (x_1x_2x_3q_1q_2q_3r_1)^{c_4} r_1{}^{c_5}, \end{aligned}$$

where parameters move in the range (3.3). Since no conditions are imposed on four parameters b_1, b_4, c_1 and c_4, we can first take out the term

$$\frac{1}{(1-q_1)(1-x_1r_1r_2)(1-x_1x_2x_3q_1q_2q_3r_1)(1-x_1x_2x_3x_4q_1q_2q_3r_1r_2)}, \tag{3.4}$$

and the remaining terms are expressed as

$$\sum (x_1q_1r_1)^{a_1} (x_1x_2q_1q_2r_1)^{a_2} (x_2r_2)^{b_2} (x_3r_2)^{b_3} (x_1q_1q_2)^{c_2} (x_1x_2q_1q_2q_3)^{c_3} r_1{}^{c_5}. \tag{3.5}$$

In the following, we calculate the sum (3.5) under the conditions (3.3) by dividing into five cases. (Note that all parameters are non-negative.)

(i) The case $b_2 \le a_1$ and $b_3 \le a_2$.

In this case, the conditions (3.3) are automatically satisfied. We put $a_1 = b_2 + d_1$, $a_2 = b_3 + d_2$ $(d_i \ge 0)$, and substitute them into (3.5). Then the expression (3.5) corresponding to this part is

(3.6)

$$\begin{aligned}
&\sum (x_1q_1r_1)^{b_2+d_1}(x_1x_2q_1q_2r_1)^{b_3+d_2}(x_2r_2)^{b_2}(x_3r_2)^{b_3}(x_1q_1q_2)^{c_2}(x_1x_2q_1q_2q_3)^{c_3}{r_1}^{c_5} \\
&= \sum (x_1x_2q_1r_1r_2)^{b_2}(x_1x_2x_3q_1q_2r_1r_2)^{b_3}(x_1q_1q_2)^{c_2}(x_1x_2q_1q_2q_3)^{c_3}{r_1}^{c_5}(x_1q_1r_1)^{d_1} \\
&\qquad \times (x_1x_2q_1q_2r_1)^{d_2} \\
&= \frac{1}{\begin{aligned}(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1x_2q_1q_2q_3)(1-x_1x_2q_1q_2r_1) \\ \times (1-x_1x_2q_1r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2).\end{aligned}}
\end{aligned}$$

(ii) The case $b_2 \le a_1$, $b_3 \ge a_2 + 1$ and $a_2 + c_5 \ge b_3$.

In this case we put $a_1 = b_2 + d_1$, $b_3 = a_2 + d_2 + 1$. Then the remaining conditions are

$$\begin{cases} c_3 \ge d_2 + 1, \\ c_5 \ge d_2 + 1. \end{cases}$$

Hence we put $c_3 = d_2 + d_3 + 1$ and $c_5 = d_2 + d_4 + 1$. Then the expression (3.5) corresponding to this part is

(3.7)

$$\begin{aligned}
&\sum (x_1q_1r_1)^{b_2+d_1}(x_1x_2q_1q_2r_1)^{a_2}(x_2r_2)^{b_2}(x_3r_2)^{a_2+d_2+1}(x_1q_1q_2)^{c_2}(x_1x_2q_1q_2q_3)^{d_2+d_3+1} \\
&\qquad \times {r_1}^{d_2+d_4+1} \\
&= \sum x_1x_2x_3q_1q_2q_3r_1r_2(x_1x_2x_3q_1q_2r_1r_2)^{a_2}(x_1x_2q_1r_1r_2)^{b_2}(x_1q_1q_2)^{c_2}(x_1q_1r_1)^{d_1} \\
&\qquad \times (x_1x_2x_3q_1q_2q_3r_1r_2)^{d_2}(x_1x_2q_1q_2q_3)^{d_3}{r_1}^{d_4} \\
&= \frac{x_1x_2x_3q_1q_2q_3r_1r_2}{\begin{aligned}(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1x_2q_1q_2q_3)(1-x_1x_2q_1r_1r_2) \\ \times (1-x_1x_2x_3q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2q_3r_1r_2).\end{aligned}}
\end{aligned}$$

(iii) The case $b_2 \le a_1$, $b_3 \ge a_2 + 1$ and $a_2 + c_5 + 1 \le b_3$.

In this case we put $a_1 = b_2 + d_1$, $b_3 = a_2 + d_2 + 1$. Then from the condition $a_2 + c_5 + 1 \le b_3$, we have $d_2 \ge c_5$. We put $d_2 = c_5 + d_3$. Then we have $b_3 = a_2 + c_5 + d_3 + 1$. The remaining conditions are

$$\begin{cases} c_3 \ge c_5 + d_3 + 1, \\ d_1 \ge d_3 + 1. \end{cases}$$

Hence we put $c_3 = c_5 + d_3 + d_4 + 1$ and $d_1 = d_3 + d_5 + 1$. And thus we have $a_1 = b_2 + d_3 + d_5 + 1$, $b_3 = a_2 + c_5 + d_3 + 1$ and $c_3 = c_5 + d_3 + d_4 + 1$. Then the expression (3.5) corresponding to this part is

(3.8)

$$\begin{aligned}
&\sum (x_1q_1r_1)^{b_2+d_3+d_5+1}(x_1x_2q_1q_2r_1)^{a_2}(x_2r_2)^{b_2}(x_3r_2)^{a_2+c_5+d_3+1}(x_1q_1q_2)^{c_2}\\
&\qquad \times (x_1x_2q_1q_2q_3)^{c_5+d_3+d_4+1}{r_1}^{c_5}\\
=&\sum {x_1}^2x_2x_3{q_1}^2q_2q_3r_1r_2(x_1x_2x_3q_1q_2r_1r_2)^{a_2}(x_1x_2q_1r_1r_2)^{b_2}(x_1q_1q_2)^{c_2}\\
&\qquad \times (x_1x_2x_3q_1q_2q_3r_1r_2)^{c_5}({x_1}^2x_2x_3{q_1}^2q_2q_3r_1r_2)^{d_3}(x_1x_2q_1q_2q_3)^{d_4}(x_1q_1r_1)^{d_5}\\
=&\frac{{x_1}^2x_2x_3{q_1}^2q_2q_3r_1r_2}{(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1x_2q_1q_2q_3)(1-x_1x_2q_1r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2)}\\
&\qquad \times (1-x_1x_2x_3q_1q_2q_3r_1r_2)(1-{x_1}^2x_2x_3{q_1}^2q_2q_3r_1r_2).
\end{aligned}$$

(iv) The case $b_2 \geq a_1+1$ and $a_2 \geq b_3$.

In this case we put $b_2 = a_1+d_1+1$, $a_2 = b_3+d_2$. Then the remaining conditions are

$$\begin{cases} c_2 \geq d_1+1, \\ c_5 \geq d_1+1. \end{cases}$$

And we put $c_2 = d_1+d_3+1$ and $c_5 = d_1+d_4+1$. Then the expression (3.5) corresponding to this part is

(3.9)

$$\begin{aligned}
&\sum (x_1q_1r_1)^{a_1}(x_1x_2q_1q_2r_1)^{b_3+d_2}(x_2r_2)^{a_1+d_1+1}(x_3r_2)^{b_3}(x_1q_1q_2)^{d_1+d_3+1}\\
&\qquad \times (x_1x_2q_1q_2q_3)^{c_3}{r_1}^{d_1+d_4+1}\\
=&\sum x_1x_2q_1q_2r_1r_2(x_1x_2q_1r_1r_2)^{a_1}(x_1x_2x_3q_1q_2r_1r_2)^{b_3}(x_1x_2q_1q_2q_3)^{c_3}(x_1x_2q_1q_2r_1r_2)^{d_1}\\
&\qquad \times (x_1x_2q_1q_2r_1)^{d_2}(x_1q_1q_2)^{d_3}{r_1}^{d_4}\\
=&\frac{x_1x_2q_1q_2r_1r_2}{(1-r_1)(1-x_1q_1q_2)(1-x_1x_2q_1q_2q_3)(1-x_1x_2q_1q_2r_1)(1-x_1x_2q_1r_1r_2)}\\
&\qquad \times (1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2).
\end{aligned}$$

(v) The case $b_2 \geq a_1+1$ and $b_3 \geq a_2+1$.

In this case we put $b_2 = a_1+d_1+1$, $b_3 = a_2+d_2+1$. Then the remaining conditions are

$$\begin{cases} c_2 \geq d_1+1, \\ c_3 \geq d_2+1, \\ c_5 \geq d_1+1, \\ c_5 \geq d_1+d_2+2. \end{cases}$$

We put $c_2 = d_1+d_3+1$, $c_3 = d_2+d_4+1$ and $c_5 = d_1+d_5+1$. Then there remains one condition $d_5 \geq d_2+1$, and we put $d_5 = d_2+d_6+1$. Thus we have

$$\begin{cases} b_2 = a_1+d_1+1, \\ b_3 = a_2+d_2+1, \\ c_2 = d_1+d_3+1, \\ c_3 = d_2+d_4+1, \\ c_5 = d_1+d_2+d_6+2. \end{cases}$$

Then the expression (3.5) corresponding to this part is

(3.10)

$$\begin{aligned}
&\sum (x_1q_1r_1)^{a_1}(x_1x_2q_1q_2r_1)^{a_2}(x_2r_2)^{a_1+d_1+1}(x_3r_2)^{a_2+d_2+1}(x_1q_1q_2)^{d_1+d_3+1}\\
&\qquad \times (x_1x_2q_1q_2q_3)^{d_2+d_4+1}{r_1}^{d_1+d_2+d_6+2}\\
=&\sum {x_1}^2{x_2}^2x_3{q_1}^2{q_2}^2q_3{r_1}^2{r_2}^2(x_1x_2q_1r_1r_2)^{a_1}(x_1x_2x_3q_1q_2r_1r_2)^{a_2}(x_1x_2q_1q_2r_1r_2)^{d_1}\\
&\qquad \times (x_1x_2x_3q_1q_2q_3r_1r_2)^{d_2}(x_1q_1q_2)^{d_3}(x_1x_2q_1q_2q_3)^{d_4}{r_1}^{d_6}\\
=&\frac{{x_1}^2{x_2}^2x_3{q_1}^2{q_2}^2q_3{r_1}^2{r_2}^2}{\begin{array}{r}(1-r_1)(1-x_1q_1q_2)(1-x_1x_2q_1q_2q_3)(1-x_1x_2q_1r_1r_2)(1-x_1x_2q_1q_2r_1r_2)\\ \times(1-x_1x_2x_3q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2q_3r_1r_2).\end{array}}
\end{aligned}$$

Finally we add the expressions (3.6) $\sim$ (3.10) and multiply (3.4). Then after some calculations we have $F_{3,2} = f/g$ where

$$\begin{aligned}
f =&\ 1-({x_1}^2x_2{q_1}^2q_2+{x_1}^2x_2x_3{q_1}^2q_2q_3+{x_1}^2{x_2}^2x_3{q_1}^2{q_2}^2q_3+{x_1}^3{x_2}^2x_3{q_1}^3{q_2}^2q_3){r_1}^2r_2\\
&+{x_1}^3{x_2}^2x_3{q_1}^3{q_2}^2q_3{r_1}^3r_2-{x_1}^3{x_2}^2x_3{q_1}^3{q_2}^2q_3{r_1}^2{r_2}^2+({x_1}^3{x_2}^2x_3{q_1}^3{q_2}^2q_3\\
&+{x_1}^4{x_2}^2x_3{q_1}^4{q_2}^2q_3+{x_1}^4{x_2}^3x_3{q_1}^4{q_2}^3q_3+{x_1}^4{x_2}^3{x_3}^2{q_1}^4{q_2}^3{q_3}^2){r_1}^3{r_2}^2\\
&-{x_1}^6{x_2}^4{x_3}^2{q_1}^6{q_2}^4{q_3}^2{r_1}^5{r_2}^3,\\
g =&\ (1-q_1)(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1r_1r_2)(1-x_1x_2q_1q_2q_3)\\
&\times(1-x_1x_2q_1q_2r_1)(1-x_1x_2q_1r_1r_2)(1-x_1x_2x_3q_1q_2q_3r_1)(1-x_1x_2x_3q_1q_2r_1r_2)\\
&\times(1-x_1x_2x_3x_4q_1q_2q_3r_1r_2)(1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2q_3r_1r_2)\\
&\times(1-{x_1}^2x_2x_3{q_1}^2q_2q_3r_1r_2).
\end{aligned}$$

By definition we have

$$\begin{aligned}
f_{110} &= {x_1}^2x_2{q_1}^2q_2+{x_1}^2x_2x_3{q_1}^2q_2q_3+{x_1}^2{x_2}^2x_3{q_1}^2{q_2}^2q_3,\\
f_{111} &= {x_1}^3{x_2}^2x_3{q_1}^3{q_2}^2q_3,\\
f_{211} &= {x_1}^4{x_2}^2x_3{q_1}^4{q_2}^2q_3+{x_1}^4{x_2}^3x_3{q_1}^4{q_2}^3q_3+{x_1}^4{x_2}^3{x_3}^2{q_1}^4{q_2}^3{q_3}^2,\\
f_{222} &= {x_1}^6{x_2}^4{x_3}^2{q_1}^6{q_2}^4{q_3}^2,
\end{aligned}$$

and hence

$$f = 1-(f_{110}+f_{111}){r_1}^2r_2+f_{111}{r_1}^3r_2-f_{111}{r_1}^2{r_2}^2+(f_{111}+f_{211}){r_1}^3{r_2}^2-f_{222}{r_1}^5{r_2}^3.$$

Therefore we obtain the desired expression for the case $m = 3$. q.e.d.

Concerning the decomposition of $\{\lambda_1, \lambda_2\}\{\mu_1, \mu_2\}$, an explicit decomposition formula is already given in Remmel-Whitehead [22]. As a corollary of Theorem 4, we give the following quite simple new decomposition formula that resembles the one in Corollary 3.

Corollary 5. *The following decomposition formula holds*:

$$\{\lambda_1, \lambda_2\}\{\mu_1, \mu_2\} = \sum_{\left\{\substack{a+d+g=\lambda_1-\lambda_2 \\ c+f+h+i=\lambda_2 \\ b+d+f=\mu_1-\mu_2 \\ e+g+h+i=\mu_2 \\ a,b,c,d,e,f,g,h,i\geq 0 \\ dh=0}\right.} \Big[(a+b)\{1,0,0,0\} + (c+d+e)\{1,1,0,0\} + (f+g)\{1,1,1,0\} + h\{2,1,1,0\} + i\{1,1,1,1\} \Big].$$

Example. We consider the case $\{32\}\{21\}$. In this case the set of integers satisfying the conditions

$$\begin{cases} a+d+g=3-2=1, \\ c+f+h+i=2, \\ b+d+f=2-1=1, \\ e+g+h+i=1, \\ a,b,c,d,e,f,g,h,i\geq 0, \\ dh=0 \end{cases}$$

is given as follows:

a	b	c	d	e	f	g	h	i	$\{\nu\}$
1	1	2	0	1	0	0	0	0	$\{53\}$
0	1	2	0	0	0	1	0	0	$\{431\}$
1	1	1	0	0	0	0	1	0	$\{521\}$
1	1	1	0	0	0	0	0	1	$\{421^2\}$
0	0	2	1	1	0	0	0	0	$\{4^2\}$
0	0	1	1	0	0	0	0	1	$\{3^21^2\}$
0	0	1	0	0	1	1	0	0	$\{3^22\}$
1	0	1	0	1	1	0	0	0	$\{431\}$
1	0	0	0	0	1	0	1	0	$\{42^2\}$
1	0	0	0	0	1	0	0	1	$\{32^21\}$

Hence we have the decomposition

$$\{32\}\{21\} = \{53\} + \{521\} + \{4^2\} + 2\{431\} + \{42^2\} + \{421^2\} + \{3^22\} + \{3^21^2\} + \{32^21\}.$$

Proof of Corollary 5. We follow a similar method as in the proof of Corollary 3. First, we have

$$
\begin{aligned}
F_{2,2} = {} & \frac{1 - {x_1}^2 x_2 {q_1}^2 q_2 {r_1}^2 r_2}{(1-q_1)(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1r_1r_2)(1-x_1x_2q_1q_2r_1)} \\
& \qquad \times (1-x_1x_2q_1r_1r_2)(1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2) \\
= {} & \sum {q_1}^a {r_1}^b (x_1q_1q_2)^c (x_1q_1r_1)^d (x_1r_1r_2)^e (x_1x_2q_1q_2r_1)^f (x_1x_2q_1r_1r_2)^g \\
& \qquad \times (x_1x_2q_1q_2r_1r_2)^h (x_1x_2x_3q_1q_2r_1r_2)^i \\
& - \sum {q_1}^a {r_1}^b (x_1q_1q_2)^c (x_1q_1r_1)^d (x_1r_1r_2)^e (x_1x_2q_1q_2r_1)^f (x_1x_2q_1r_1r_2)^g \\
& \qquad \times (x_1x_2q_1q_2r_1r_2)^h (x_1x_2x_3q_1q_2r_1r_2)^i ({x_1}^2 x_2 {q_1}^2 q_2 {r_1}^2 r_2) \\
= {} & \sum {x_1}^{c+d+e+f+g+h+i} {x_2}^{f+g+h+i} {x_3}^i {q_1}^{a+c+d+f+g+h+i} {q_2}^{c+f+h+i} {r_1}^{b+d+e+f+g+h+i} {r_2}^{e+g+h+i} \\
& - \sum {x_1}^{c+d+e+f+g+h+i+2} {x_2}^{f+g+h+i+1} {x_3}^i {q_1}^{a+c+d+f+g+h+i+2} {q_2}^{c+f+h+i+1} \\
& \qquad \times {r_1}^{b+d+e+f+g+h+i+2} {r_2}^{e+g+h+i+1}.
\end{aligned}
$$

The first line in the last expression corresponds to the partition

$$\{*, c+d+e+f+g+h+i, f+g+h+i, i\}$$

in $\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\} = \{a+c+d+f+g+h+i, c+f+h+i\}\{b+d+e+f+g+h+i, e+g+h+i\}$. It is easy to see that the above $*$ part is equal to $a+b+c+d+e+f+g+2h+i$. Hence, the first line is equal to

$$
\sum_{\begin{cases} a+d+g=\lambda_1-\lambda_2 \\ c+f+h+i=\lambda_2 \\ b+d+f=\mu_1-\mu_2 \\ e+g+h+i=\mu_2 \\ a,b,c,d,e,f,g,h,i \geq 0 \end{cases}} \Big[(a+b)\{1,0,0,0\} + (c+d+e)\{1,1,0,0\} + (f+g)\{1,1,1,0\} + h\{2,1,1,0\} + i\{1,1,1,1\} \Big].
$$

In the similar way, the second line corresponds to the partitions

$$\{a+b+c+d+e+f+g+2h+i+3, c+d+e+f+g+h+i+2, f+g+h+i+1, i\}$$

in $\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\} = \{a+c+d+f+g+h+i+2, c+f+h+i+1\}\{b+d+e+f+g+h+i+2, e+g+h+i+1\}$. In this case, the above partition can be expressed as

$$
\begin{aligned}
& (a+b)\{1,0,0,0\} + (c+d+e)\{1,1,0,0\} + (f+g)\{1,1,1,0\} \\
& \qquad + h\{2,1,1,0\} + i\{1,1,1,1\} + \{3,2,1,0\} \\
= {} & (a+b)\{1,0,0,0\} + (c+(d+1)+e)\{1,1,0,0\} + (f+g)\{1,1,1,0\} \\
& \qquad + (h+1)\{2,1,1,0\} + i\{1,1,1,1\}.
\end{aligned}
$$

In addition, we have

$$\begin{cases} a+(d+1)+g=\lambda_1-\lambda_2, \\ c+f+(h+1)+i=\lambda_2, \\ b+(d+1)+f=\mu_1-\mu_2, \\ e+g+(h+1)+i=\mu_2. \end{cases}$$

Hence, by changing the variables d, h, we know that the second line is equal to

$$\sum_{\begin{cases} a+d+g=\lambda_1-\lambda_2 \\ c+f+h+i=\lambda_2 \\ b+d+f=\mu_1-\mu_2 \\ e+g+h+i=\mu_2 \\ a,b,c,e,f,g,i\geq 0,\ d,h\geq 1 \end{cases}} \Big[(a+b)\{1,0,0,0\}+(c+d+e)\{1,1,0,0\}+(f+g)\{1,1,1,0\} + h\{2,1,1,0\}+i\{1,1,1,1\} \Big].$$

Subtracting this expression from the first line, we obtain the desired result. q.e.d.

Remark. From Corollary 5, we know that the number of irreducible components of $\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\}$ is equal to the number of lattice points in $\mathbf{R}^5$ with coordinate (d,f,g,h,i) satisfying the following conditions:

$$\begin{cases} d+g\leq\lambda_1-\lambda_2, \\ f+h+i\leq\lambda_2, \\ d+f\leq\mu_1-\mu_2, \\ g+h+i\leq\mu_2, \\ d,f,g,h,i\geq 0, \\ dh=0. \end{cases}$$

In general it is hard to express this number explicitly as a polynomial of λ_i and μ_i. As one example, in case $0\leq\mu_2\leq\lambda_2\leq\mu_1-\mu_2\leq\mu_1\leq\lambda_1-\lambda_2$, we can directly verify that the number of irreducible components of $\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\}$ is equal to

$$(\lambda_2\mu_1+\mu_1-\mu_2+1)\binom{\mu_2+2}{2}-\binom{\lambda_2}{2}\binom{\mu_2+2}{2}-(\lambda_2+\mu_1)\binom{\mu_2+2}{3}+\binom{\mu_2+2}{4}.$$

4. Conjectures.

Generation functions stated in Theorem 4 possess several properties. In this section we summarize these properties in the form of conjectures on $F_{m,2}$ (Conjecture 1), and in addition, give the explicit conjectural expression of the quite lengthy next generating function $F_{5,2}$ (Conjecture 2). We use the same notations as in the previous section.

Conjecture 1. *$F_{m,2}$ can be expressed as a ratio of two polynomials*:

$$\frac{f(x_1q_1,\cdots,x_mq_m,r_1,r_2)}{g(x_1,\cdots,x_{m+1},q_1,\cdots,q_m,r_1,r_2)}.$$

Here, the denominator g is given by

$$g(x_1,\cdots,x_{m+1},q_1,\cdots,q_m,r_1,r_2)=\prod_{\begin{cases}0\le i\le m+1\\ 0\le j\le m\\ 0\le k\le 2\\ j+k=i+1\end{cases}}(1-x_1\cdots x_iq_1\cdots q_jr_1\cdots r_k)$$
$$\times\prod_{2\le i+2\le j\le m}\{1-(x_1q_1)^2\cdots(x_iq_i)^2(x_{i+1}q_{i+1})\cdots(x_jq_j)r_1r_2\}$$

and the numerator f can be expressed as a linear combination of $f_{a_1\cdots a_m}{r_1}^p{r_2}^q$ ($a_1\ge\cdots\ge a_m\ge 0$ and $p\ge q\ge 0$) with integer coefficients. The numerator f satisfies the following properties:

(i) *If the term $f_{a_1\cdots a_m}{r_1}^p{r_2}^q$ appears in f, then the exponent (p,q) moves in the range*

$$0\le p\le\frac{(m-1)(m+2)}{2},\quad 0\le q\le\frac{m(m-1)}{2},\quad p-q\le m-1,$$
$$\max\{p,p+q-(m-1)\}\le\sum_{i=1}^m a_i\le\min\{p+q,p+\frac{(m-1)(m-2)}{2}\}.$$

(ii) *The coefficients of ${r_2}^0$ and ${r_2}^1$ in f are given by*

$$1-\sum_{k=2}^{m-1}(-1)^k(f_{\underbrace{1\cdots1}_{k}\underbrace{0\cdots0}_{m-k}}+f_{\underbrace{1\cdots1}_{k+1}\underbrace{0\cdots0}_{m-k-1}})\,{r_1}^k r_2-(-1)^m f_{\underbrace{1\cdots1}_{m}}\,{r_1}^m r_2.$$

(iii) *The coefficient of ${r_1}^2{r_2}^2$ in f is given by*

$$-(f_{111\underbrace{0\cdots0}_{m-3}}+f_{1111\underbrace{0\cdots0}_{m-4}})\,{r_1}^2{r_2}^2.$$

(iv) *The coefficient of ${r_1}^{\frac{(m-1)(m+2)}{2}}{r_2}^{\frac{m(m-1)}{2}}$, which is the last term of f, is given by*

$$(-1)^{\frac{m(m-1)}{2}}f_{\underbrace{m-1,\cdots,m-1}_{m}}\,{r_1}^{\frac{(m-1)(m+2)}{2}}{r_2}^{\frac{m(m-1)}{2}}.$$

(v) *The following reciprocal property holds*:

$$\begin{aligned}&f(x_1q_1,\cdots,x_mq_m,r_1,r_2)\\&\quad=(-1)^{\frac{m(m-1)}{2}}(x_1q_1)^{m(m-1)}(x_2q_2)^{(m-1)^2}\cdots(x_{m-1}q_{m-1})^{2(m-1)}(x_mq_m)^{m-1}\\&\qquad\times{r_1}^{\frac{(m-1)(m+2)}{2}}{r_2}^{\frac{m(m-1)}{2}}f\left(\frac{1}{x_1q_1},\cdots,\frac{1}{x_mq_m},\frac{1}{r_1},\frac{1}{r_2}\right).\end{aligned}$$

(vi) *If we set $x_mq_m=0$ in f, then the polynomial $f(x_1q_1,\cdots,x_{m-1}q_{m-1},0,r_1,r_2)$ gives the numerator of $F_{m-1,2}$.*

Remark that the parameter x_{m+1} appears only in the denominator of $F_{m,2}$. If we put $r_2 = 0$ in $F_{m,2}$, then we just obtain the generating function $F_{m,1}$ appeared in Theorem 1. Unfortunately, the numerator f cannot be uniquely characterized by only the above properties.

It is easy to see that the reciprocal property (v) in Conjecture 1 is equivalent to say that the term $kf_{a_1\cdots a_m}{r_1}^p{r_2}^q$ ($k \in \mathbf{Z}$) appears in the numerator f if and only if the term $(-1)^{\frac{m(m-1)}{2}}kf_{(m-1)-a_m,(m-1)-a_{m-1},\cdots,(m-1)-a_1}{r_1}^{\frac{(m-1)(m+2)}{2}-p}{r_2}^{\frac{m(m-1)}{2}-q}$ appears in f. (Remark that $f_{0\cdots 0} = 1$.)

We can easily see that Conjecture 1 actually holds for the case $m \leq 4$.

For the next case $m = 5$, it seems hard to obtain the explicit formula by the method stated in the proof of Theorem 4. But by using computers, we now arrive at the following conjecture.

Conjecture 2. *In case $m = 5$, the numerator of the generating function is given as follows*:

$$
\begin{aligned}
&f(x_1q_1, \cdots, x_5q_5, r_1, r_2) = \\
&\quad 1 - (f_{11000} + f_{11100}){r_1}^2r_2 + (f_{11100} + f_{11110}){r_1}^3r_2 - (f_{11110} + f_{11111}){r_1}^4r_2 + f_{11111}{r_1}^5r_2 \\
&\quad - (f_{11100} + f_{11110}){r_1}^2{r_2}^2 + (f_{11100} + f_{21100} + 5f_{11110} + f_{21110} + 5f_{11111}){r_1}^3{r_2}^2 \\
&\quad - (f_{11110} + f_{21110} + 6f_{11111} + f_{21111}){r_1}^4{r_2}^2 + (f_{11111} + f_{21111}){r_1}^5{r_2}^2 + (f_{11110} + f_{21110} \\
&\quad + 5f_{11111} + f_{21111}){r_1}^3{r_2}^3 - (f_{11110} + f_{21110} + 6f_{11111} + f_{31110} + f_{22110} + 6f_{21111} + f_{31111} \\
&\quad + f_{22111}){r_1}^4{r_2}^3 + (f_{11111} - f_{22200} - f_{22110} - f_{21111} + f_{31111} - f_{22210} - f_{22111} - f_{22220} \\
&\quad - f_{22211}){r_1}^5{r_2}^3 + (f_{22210} + 2f_{22111} + f_{22220} + 2f_{22211} + f_{22221}){r_1}^6{r_2}^3 - (f_{22211} + f_{22221}) \\
&\quad {r_1}^7{r_2}^3 - (f_{11111} + f_{21111} + f_{31111} + f_{22111}){r_1}^4{r_2}^4 + (f_{11111} + f_{21111} + f_{31111} - f_{22210} \\
&\quad - f_{22111} + f_{41111} + f_{32111} - f_{22220} - f_{22211} - f_{22221}){r_1}^5{r_2}^4 + (f_{22210} + 2f_{22111} + f_{32210} \\
&\quad + 2f_{32111} + 5f_{22220} + 10f_{22211} + f_{32220} + 2f_{32211} + 10f_{22221} + f_{32221} + 5f_{22222}){r_1}^6{r_2}^4 \\
&\quad - (f_{22220} + 2f_{22211} + f_{32220} + 2f_{32211} + 10f_{22221} + 2f_{32221} + 10f_{22222} + f_{32222}){r_1}^7{r_2}^4 \\
&\quad + (f_{22221} + f_{32221} + 5f_{22222} + f_{32222}){r_1}^8{r_2}^4 - (f_{22220} + f_{22211} + f_{22221} + f_{22222}){r_1}^5{r_2}^5 \\
&\quad + (f_{22220} + 2f_{22211} + f_{32220} + 2f_{32211} + 10f_{22221} + 2f_{32221} + 10f_{22222} + f_{32222}){r_1}^6{r_2}^5 \\
&\quad - (f_{22211} + f_{32220} + 2f_{32211} + 10f_{22221} + f_{42211} + f_{33220} + 2f_{33211} + 10f_{32221} + 30f_{22222} \\
&\quad + f_{42221} + 2f_{33221} + 10f_{32222} + f_{33222}){r_1}^7{r_2}^5 + (f_{22221} + 10f_{22222} + 2f_{32221} + 10f_{32222} \\
&\quad + 2f_{33221} + f_{42221} + 2f_{33222} + f_{42222}){r_1}^8{r_2}^5 - (f_{42222} + f_{33222} + f_{32222} + f_{22222}){r_1}^9{r_2}^5 \\
&\quad + (f_{32222} + f_{32221} + 5f_{22222} + f_{22221}){r_1}^6{r_2}^6 - (f_{42222} + 2f_{33222} + f_{42221} + 2f_{33221} \\
&\quad + 10f_{32222} + 2f_{32221} + 10f_{22222} + f_{22221}){r_1}^7{r_2}^6 + (f_{43222} + 2f_{33322} + f_{43221} + 2f_{33321} \\
&\quad + 5f_{42222} + 10f_{33222} + f_{42221} + 2f_{33221} + 10f_{32222} + f_{32221} + 5f_{22222}){r_1}^8{r_2}^6 + (f_{33333} \\
&\quad + f_{33332} + f_{33331} - f_{43222} - f_{33322} + f_{33330} + f_{33321} - f_{42222} - f_{33222} - f_{32222}){r_1}^9{r_2}^6 \\
&\quad - (f_{33333} + f_{33332} + f_{33331} + f_{33322}){r_1}^{10}{r_2}^6 - (f_{33222} + f_{32222}){r_1}^7{r_2}^7 + (f_{43222} + 2f_{33322} \\
&\quad + f_{42222} + 2f_{33222} + f_{32222}){r_1}^8{r_2}^7 + (f_{33333} - f_{44222} - f_{43322} - f_{33332} + f_{33331} - f_{43222}
\end{aligned}
$$

$$- f_{33322} - f_{42222} - f_{33222}){r_1}^9{r_2}^7 - (f_{43333} + f_{43332} + 6f_{33333} + f_{43331} + f_{43322} + 6f_{33332} + f_{33331} + f_{33322}){r_1}^{10}{r_2}^7 + (f_{43333} + f_{43332} + 5f_{33333} + f_{33332}){r_1}^{11}{r_2}^7 + (f_{33333} + f_{33332}) {r_1}^9{r_2}^8 - (f_{43333} + f_{43332} + 6f_{33333} + f_{33332}){r_1}^{10}{r_2}^8 + (f_{44333} + f_{44332} + 5f_{43333} + f_{43332} + 5f_{33333}){r_1}^{11}{r_2}^8 - (f_{44333} + f_{43333}){r_1}^{12}{r_2}^8 - f_{33333}{r_1}^9{r_2}^9 - (f_{43333} + f_{33333}){r_1}^{10}{r_2}^9 + (f_{44333} + f_{43333}){r_1}^{11}{r_2}^9 - (f_{44433} + f_{44333}){r_1}^{12}{r_2}^9 + f_{44444}{r_1}^{14}{r_2}^{10}.$$

This numerator actually consists of 3700 monomials if we express it as a polynomial of x_i, q_i and r_i. The coefficients appeared in this numerator are limited to 1, 2, 5, 6, 10, 30, and they may have some combinatorial meaning. It can be directly verified that the numerator f in this conjecture satisfies all properties stated in Conjecture 1.

5. Littlewood-Richardson polynomials.

We fix three partitions $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$ such that $|\lambda| + |\mu| = |\nu|$. Then it is known that for non-negative integers N, the value $c^{N\nu}_{N\lambda\, N\mu}$ can be expressed as a polynomial of N, and this polynomial is now called the Littlewood-Richardson polynomial (cf. [9], [18]). By using the method of hives (cf. Buch [6], or Knutson-Tao [12]), we can easily see that the degree of the Littlewood-Richardson polynomial is at most $(m-1)(n-1)$ for the case $\{\lambda\} = \{\lambda_1, \cdots, \lambda_m\}$ and $\{\mu\} = \{\mu_1, \cdots, \mu_n\}$. (In fact in this case, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of lattice points in some $(m-1)(n-1)$-dimensional convex polytope whose boundaries are determined by the conditions on hives.) In this section, we give a generating function of the Littlewood-Richardson polynomial for the case $\{\lambda\} = \{\lambda_1, \lambda_2\}$ and $\{\mu\} = \{\mu_1, \mu_2\}$, as an application of Theorem 4.

First we define a polynomial $f^{\nu}_{\lambda\mu}(N)$ by

$$f^{\nu}_{\lambda\mu}(N) = \begin{cases} c^{N\nu}_{N\lambda\, N\mu} & (c^{\nu}_{\lambda\mu} \neq 0), \\ 0 & (c^{\nu}_{\lambda\mu} = 0). \end{cases}$$

And next, define a generating function $LR_{2,2}(N)$ by

$$LR_{2,2}(N) = \sum_{\substack{\{\lambda\} = \{\lambda_1, \lambda_2\} \\ \{\mu\} = \{\mu_1, \mu_2\} \\ \{\nu\} = \{\nu_1, \nu_2, \nu_3, \nu_4\} \\ |\lambda| + |\mu| = |\nu|}} f^{\nu}_{\lambda\mu}(N)\, {x_1}^{\nu_2}{x_2}^{\nu_3}{x_3}^{\nu_4}{q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}.$$

Namely, the coefficient of ${x_1}^{\nu_2}{x_2}^{\nu_3}{x_3}^{\nu_4}{q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}$ in $LR_{2,2}(N)$ gives the Littlewood-Richardson polynomial for $\{\lambda\} = \{\lambda_1, \lambda_2\}$, $\{\mu\} = \{\mu_1, \mu_2\}$ and $\{\nu\} = \{\nu_1, \cdots, \nu_4\}$. Then we have the following theorem.

Theorem 6. *The generating function $LR_{2,2}(N)$ is given by the following form:*

$$\frac{(1 - {x_1}^2x_2{q_1}^2q_2{r_1}^2r_2)(1 + {x_1}^2x_2{q_1}^2q_2{r_1}^2r_2(N-1))}{\begin{array}{l}(1-q_1)(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1r_1r_2)(1-x_1x_2q_1q_2r_1) \\ \qquad \times (1-x_1x_2q_1r_1r_2)(1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2).\end{array}}$$

As an example, by using computers we know that the coefficient of ${x_1}^7{x_2}^3{q_1}^7{q_2}^4{r_1}^6{r_2}^3$ in $LR_{2,2}(N)$ is $3N+1$, which gives the Littlewood-Richardson polynomial for the case $\{\lambda\}=\{74\}$, $\{\mu\}=\{63\}$ and $\{\nu\}=\{10,73\}$. Note that if we put $N=1$ into $LR_{2,2}(N)$, then we just obtain the generating function $F_{2,2}$ in Theorem 4.

Since the degree of the Littlewood-Richardson polynomial is at most 1 for the case $\{\lambda\}=\{\lambda_1,\lambda_2\}$ and $\{\mu\}=\{\mu_1,\mu_2\}$, it suffices to know the function $LR_{2,2}(0)$ in order to prove Theorem 6. In fact, using the function $F_{2,2}$, $LR_{2,2}(N)$ can be expressed as

$$LR_{2,2}(N)=(F_{2,2}-LR_{2,2}(0))N+LR_{2,2}(0).$$

The generating function $LR_{2,2}(0)$ may be considered as a support function of $c^{\nu}_{\lambda\mu}$. Namely,

$$c^{\nu}_{\lambda\mu}>0 \Longleftrightarrow \text{the term } {x_1}^{\nu_2}{x_2}^{\nu_3}{x_3}^{\nu_4}{q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2} \text{ appears in } LR_{2,2}(0).$$

(By definition, the coefficient of each term in $LR_{2,2}(0)$ is always 1.) The generating function $LR_{2,2}(0)$ is given in the following form. Theorem 6 follows immediately from this proposition.

Proposition 7.

$$LR_{2,2}(0)=\frac{(1-{x_1}^2x_2{q_1}^2q_2{r_1}^2r_2)^2}{\begin{array}{r}(1-q_1)(1-r_1)(1-x_1q_1q_2)(1-x_1q_1r_1)(1-x_1r_1r_2)(1-x_1x_2q_1q_2r_1)\\ \times(1-x_1x_2q_1r_1r_2)(1-x_1x_2q_1q_2r_1r_2)(1-x_1x_2x_3q_1q_2r_1r_2).\end{array}}$$

It is surprising that all coefficients in this (somewhat complicated) formal power series are 1. If we put $x_1=x_2=x_3=1$ into the above $LR_{2,2}(0)$, we obtain the generating function expressing the number of different "types" of partitions appearing in the tensor product $\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\}$. For example, it is expanded as

$$1+q_1+r_1+{q_1}^2+q_1q_2+{r_1}^2+r_1r_2+2q_1r_1+\cdots\cdots+13{q_1}^5q_2{r_1}^3{r_2}^2+\cdots\cdots,$$

and hence we know that 13 different types of partitions appear in the product $\{51\}\{32\}$.

We can prove Proposition 7 by a similar method stated in the proof of Theorem 4, and we here state only the outline of the proof. First, by using the Littlewood-Richardson rule (or hives), we can easily show that $c^{\nu}_{\lambda\mu}>0$ if and only if

$$\begin{aligned}
&|\lambda|+|\mu|=|\nu|,\\
&\nu_4\le\min\{\lambda_2,\mu_2\},\quad \max\{\lambda_2,\mu_2\}\le\nu_2,\\
&\nu_3\le\min\{\lambda_1,\mu_1\},\quad \max\{\lambda_1,\mu_1\}\le\nu_1,\\
&\nu_3+\nu_4\le\lambda_2+\mu_2\le\nu_2+\nu_3\le\lambda_1+\mu_1,\\
&\nu_2+\nu_4\le\min\{\lambda_1+\mu_2,\lambda_2+\mu_1\}.
\end{aligned}$$

Under these conditions, we calculate the sum

$$\sum_{\lambda,\mu,\nu} {x_1}^{\nu_2}{x_2}^{\nu_3}{x_3}^{\nu_4}{q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}.$$

Then after many case by case examinations, we can finally obtain the expression stated in Proposition 7.

Finally, we give two applications of our formula on generating functions. The first one concerns the multiplicity-free product. In [26] Stembridge gave a classification of the pair (λ, μ) such that the product $\{\lambda\}\{\mu\}$ is multiplicity-free. We here give another proof of this result for the case $\{\lambda\} = \{\lambda_1, \lambda_2\}$ and $\{\mu\} = \{\mu_1, \mu_2\}$, by using our generating function $F_{2,2}$ and $LR_{2,2}(0)$.

Theorem 8 ([26]). *The product $\{\lambda_1, \lambda_2\}\{\mu_1, \mu_2\}$ is multiplicity-free if and only if $\{\lambda_1, \lambda_2\}$ or $\{\mu_1, \mu_2\}$ forms a rectangle.*

Proof. It is clear that the product $\{\lambda_1, \lambda_2\}\{\mu_1, \mu_2\}$ is multiplicity-free if and only if the coefficient of ${x_1}^{\nu_2}{x_2}^{\nu_3}{x_3}^{\nu_4}{q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}$ in $F_{2,2}$ is 1 for any $\{\nu\}$. Since the coefficient of each term in $LR_{2,2}(0) = (1 - {x_1}^2x_2{q_1}^2q_2{r_1}^2r_2)F_{2,2}$ is always 1, the above condition is equivalent to say that the coefficient of ${q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}$ in $F_{2,2}$ and in $(1 - {x_1}^2x_2{q_1}^2q_2{r_1}^2r_2)F_{2,2}$ coincides. If the term containing ${q_1}^{\lambda_1-2}{q_2}^{\lambda_2-1}{r_1}^{\mu_1-2}{r_2}^{\mu_2-1}$ appears in $F_{2,2}$, then ${q_1}^{\lambda_1}{q_2}^{\lambda_2}{r_1}^{\mu_1}{r_2}^{\mu_2}$ does not possess this property. And the converse is also true. Clearly the term containing ${q_1}^{\lambda_1-2}{q_2}^{\lambda_2-1}{r_1}^{\mu_1-2}{r_2}^{\mu_2-1}$ does not appear in $F_{2,2}$ if and only if $\{\lambda_1, \lambda_2\}$ or $\{\mu_1, \mu_2\}$ is a rectangle, and thus we complete the proof of Theorem 8. q.e.d.

Next, as the second application, we give the explicit expression of the Littlewood-Richardson polynomial. We can easily see that the generating function $LR_{2,2}(N)$ also can be expressed as

$$LR_{2,2}(N) = LR_{2,2}(0) + {x_1}^2x_2{q_1}^2q_2{r_1}^2r_2F_{2,2} \times N.$$

By using this equality, we can easily show the following theorem.

Theorem 9. *Assume that the partitions $\{\lambda\} = \{\lambda_1, \lambda_2\}$, $\{\mu\} = \{\mu_1, \mu_2\}$, $\{\nu\} = \{\nu_1, \cdots, \nu_4\}$ satisfy $c^{\nu}_{\lambda\mu} \neq 0$, and we put $\{\overline{\lambda}\} = \{\lambda_1 - 2, \lambda_2 - 1\}$, $\{\overline{\mu}\} = \{\mu_1 - 2, \mu_2 - 1\}$, $\{\overline{\nu}\} = \{\nu_1 - 3, \nu_2 - 2, \nu_3 - 1, \nu_4\}$. Then the Littlewood-Richardson polynomial corresponding to the partitions $\{\lambda\}$, $\{\mu\}$ and $\{\nu\}$ is given by $c^{\overline{\nu}}_{\overline{\lambda}\,\overline{\mu}}N + 1$. (If $\{\overline{\lambda}\}$, $\{\overline{\mu}\}$ or $\{\overline{\nu}\}$ does not form a partition, we consider $c^{\overline{\nu}}_{\overline{\lambda}\,\overline{\mu}} = 0$.)*

The essence of the proof is almost same to that of Theorem 8, and we leave the proof to the readers.

In contrast to the above theorem, it is in general hard to express the Littlewood-Richardson polynomial as an explicit function of $\{\lambda\}$, $\{\mu\}$ and $\{\nu\}$ because it requires quite complicated case by case verification. For example, assume that the partitions $\{\lambda\} = \{\lambda_1, \lambda_2, \lambda_3\}$, $\{\mu\} = \{\mu_1, \mu_2\}$ and $\{\nu\} = \{\nu_1, \cdots, \nu_5\}$ satisfy the following conditions:

$$\begin{cases} \nu_4 \leq \lambda_3 \leq \lambda_2 \leq \nu_3 \leq \nu_2 \leq \lambda_1 \leq \nu_1, \\ \lambda_2 + \mu_1 + 1 \leq \nu_2 + \nu_4, \\ \lambda_3 + \mu_2 \leq \nu_3 + \nu_4 \leq \min\{\lambda_2 + \mu_2, \lambda_3 + \mu_1\}, \\ \nu_2 + \nu_5 \leq \min\{\lambda_2 + \mu_2, \lambda_3 + \mu_1\}, \\ \lambda_3 + \nu_1 \leq \lambda_1 + \lambda_2. \end{cases}$$

Then after some calculations, we know that the Littlewood-Richardson polynomial for this case is given by

$$\begin{aligned} &\frac{1}{2}\{(-2\lambda_1 - \lambda_3 - \mu_1 + 2\nu_1 + \nu_2 + \nu_5)N + 2\}\{(\lambda_3 + \mu_1 - \nu_2 - \nu_5)N + 1\} \\ &\quad + (\lambda_1 + \lambda_2 - \lambda_3 - \nu_1)N\{(\nu_1 - \lambda_1)N + 1\} \\ &\quad + \frac{1}{2}(\lambda_3 + \mu_1 - \nu_3 - \nu_4)N\{(-2\lambda_1 - \lambda_3 - \mu_1 + 2\nu_1 + \nu_3 + \nu_4)N + 1\}. \end{aligned}$$

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