A LOWER BOUND FOR THE CLASS NUMBER OF $P^n(\boldsymbol{C})$ AND $P^n(\boldsymbol{H})$

YOSHIO AGAOKA AND EIJI KANEDA

Abstract. We obtain new lower bounds on the codimension of local isometric imbeddings of complex and quaternion projective spaces. We show that any open set of the complex projective space $P^n(\boldsymbol{C})$ (resp. quaternion projective space $P^n(\boldsymbol{H})$) cannot be locally isometrically imbedded into the euclidean space of dimension $4n-3$ (resp. $8n-4$). These estimates improve the previously known results obtained in [2] and [7].

1. Introduction

Let M be a Riemannian manifold. As is known, M can be locally or globally isometrically imbedded into a euclidean space of sufficiently large dimension (see Janet [19], Cartan [14], Nash [24], Greene–Jacobowitz [16], Gromov–Rokhlin [17]). It is a natural and interesting question to ask the least dimension of euclidean spaces into which M can be locally or globally isometrically imbedded. In this paper we will investigate the problem of local isometric imbeddings of the projective spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ and give a new estimate on the least dimension of the ambient euclidean spaces.

Let $x \in M$. Assume that there is a neighborhood U of x in M such that U is isometrically imbedded into a euclidean space $\boldsymbol{R}^D$. If any neighborhood of x cannot be isometrically imbedded into $\boldsymbol{R}^{D-1}$, then the codimension $D - \dim M$ is called the *class number* of M at x and is denoted by $\mathrm{class}(M)_x$.

Let G/K be a Riemannian symmetric space. By homogeneity, the class number of G/K is constant everywhere on G/K, which is denoted by $\mathrm{class}(G/K)$. In Agaoka–Kaneda [4], [5], [7], [8], [9] and [10] we have tried to estimate $\mathrm{class}(G/K)$ from below. In doing this we mainly used the following inequality

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K),$$

where $p(G/K)$ is the pseudo-nullity of G/K (see §2 below or [4]). For the following Riemannian symmetric spaces G/K our estimates just hit $\mathrm{class}(G/K)$, i.e., $\mathrm{class}(G/K) = \dim G/K - p(G/K)$:

a) The sphere S^n $(n \geq 2)$;
b) $CI : Sp(n)/U(n)$ $(n \geq 1)$ (see [4]);
c) The symplectic group $Sp(n)$ $(n \geq 1)$ (see [5]).

2000 *Mathematics Subject Classification.* 53B25, 53C35, 17B20.
Key words and phrases. Curvature invariant, isometric imbedding, complex projective space, quaternion projective space, root space decomposition.

As for the class numbers of the projective spaces such as the complex projective space $P^n(\boldsymbol{C})$, the quaternion projective space $P^n(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$, the following are known:

(1) $\mathrm{class}(P^n(\boldsymbol{C})) \geq \max\{n+1, [\frac{6}{5}n]\}$ $(n \geq 2)$ (see [2] and [7]);
(2) $\mathrm{class}(P^n(\boldsymbol{H})) \geq \min\{4n-3, 3n+1\}$ $(n \geq 3)$ (see [7]);
(3) $\mathrm{class}(P^n(\boldsymbol{C})) \leq n^2$ $(n \geq 2)$; $\mathrm{class}(P^n(\boldsymbol{H})) \leq 2n^2 - n$ $(n \geq 2)$ (see [22]);
(4) $\mathrm{class}(P^2(\boldsymbol{H})) = 6$; $\mathrm{class}(P^2(\boldsymbol{Cay})) = 10$ (see [8] and [22]).

It should be noted that any local isometric imbedding of $P^2(\boldsymbol{H})$ (resp. $P^2(\boldsymbol{Cay})$) into the euclidean space $\boldsymbol{R}^{14}$ (resp. $\boldsymbol{R}^{26}$) is rigid in the strongest sense (see [9] and [10]).

In this paper we will propose a new type of estimate and by applying it we will prove

Theorem 1. *Let G/K denote the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 3)$ or the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 3)$. Define an integer $q(G/K)$ by*

$$q(G/K) = \begin{cases} 4n-2, & \text{if } G/K = P^n(\boldsymbol{C})\ (n \geq 3); \\ 8n-3, & \text{if } G/K = P^n(\boldsymbol{H})\ (n \geq 3). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the euclidean space $\boldsymbol{R}^D$ with $D \leq q(G/K) - 1$. In other words,

$$\mathrm{class}(P^n(\boldsymbol{C})) \geq 2n-2 \ (n \geq 3); \quad \mathrm{class}(P^n(\boldsymbol{H})) \geq 4n-3 \ (n \geq 3).$$

It is clearly seen that Theorem 1 improves the estimates (1) and (2) stated above. However, we have to recognize a large gap between our estimate and the upper bound stated in (3), which cannot be filled at present.

Throughout this paper we will assume the differentiability of class C^∞. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [18].

2. The Gauss equation

Let M be a Riemannian manifold and g be the Riemannian metric of M. We denote by R the Riemannian curvature tensor of type $(1,3)$ with respect to g.

For each $x \in M$ we denote by $T_x(M)$ (resp. $T_x^*(M)$) the tangent (resp. cotangent) vector space of M at $x \in M$. Let r be a non-negative integer. We define a quadratic equation with respect to an unknown $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ by

$$-g(R(X,Y)Z,W) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W) \big\rangle - \big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z) \big\rangle, \tag{2.1}$$

where X, Y, Z, $W \in T_x(M)$ and $\langle \, , \, \rangle$ is the standard inner product of $\boldsymbol{R}^r$. We call (2.1) the *Gauss equation in codimension* r at x. It is well-known that for a sufficiently large r the Gauss equation (2.1) in codimension r admits a solution (see Berger [12], Berger–Bryant–Griffith [13]). On the other hand, in general, for a small r (2.1) does not admit any solution. By $\mathrm{Crank}(M)_x$ we denote the least value of r with which (2.1) admits a solution and call it the *curvature rank* of M at x. It should be noted that $\mathrm{Crank}(M)_x$ is a curvature invariant, i.e., it can be determined only by the curvature R of M at x.

As is well-known, if there is an isometric immersion $\boldsymbol{f}$ of M into $\boldsymbol{R}^D$, then the second fundamental form of $\boldsymbol{f}$ at x satisfies the Gauss equation in codimension $r = D - \dim M$. Therefore, we have

Lemma 2. $\mathrm{class}(M)_x \geq \mathrm{Crank}(M)_x$ *holds for any* $x \in M$.

In the following, we assume that $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ is a solution of the Gauss equation in codimension r. Let $X \in T_x(M)$. We define a linear mapping $\boldsymbol{\Psi}_X \colon T_x(M) \longrightarrow \boldsymbol{R}^r$ by $\boldsymbol{\Psi}_X(Y) = \boldsymbol{\Psi}(X,Y)\,(Y \in T_x(M))$. The kernel of this map $\boldsymbol{\Psi}_X$ is denoted by $\mathbf{Ker}(\boldsymbol{\Psi}_X)$. Then we can easily show the following

Lemma 3. *Let* $X \in T_x(M)$. *Then* $R(\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathbf{Ker}(\boldsymbol{\Psi}_X))X = 0$.

For the proof, see [4]. By this lemma we can get the following estimate for $\mathrm{Crank}(M)_x$: Let $X \in T_x(M)$. By $d(X)$ we denote the maximum value of the dimensions of those subspaces $V \subset T_x(M)$ such that $R(V,V)X = 0$. Then by Lemma 3 it is easily seen that $d(X) \geq \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq \dim M - r$. Set $p_M(x) = \min\{d(X) \mid X \in T_x(M)\}$. Then $p_M(x) \geq \dim M - r$, i.e., $r \geq \dim M - p_M(x)$. The number $p_M(x)$ thus defined is also a curvature invariant, which is called the *pseudo-nullity* of M at x. By the above discussion we have

Lemma 4. $\mathrm{Crank}(M)_x \geq \dim M - p_M(x)$.

In the case where M is a Riemannian homogeneous space G/K, the class number, the curvature rank and the pseudo-nullity of G/K are constant everywhere on G/K, which are denoted by $\mathrm{class}(G/K)$, $\mathrm{Crank}(G/K)$ and $p(G/K)$, respectively. Combining Lemma 4 with Lemma 2, we obtain

Proposition 5. *Let* G/K *be a Riemannian homogeneous space. Then*:

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K).$$

This is a fundamental tool in our works [5] and [7] to estimate the class numbers of Riemannian symmetric spaces from below.

Now, we show a new type of estimate:

Theorem 6. *Let* $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ *be a solution of the Gauss equation in codimension* r. *Assume that there are tangent vectors* X, $Y \in T_x(M)$ *and a subspace* $\boldsymbol{U}$ *of* $T_x(M)$ *satisfying*

(1) $\boldsymbol{\Psi}(X,Y) = 0$;
(2) $\boldsymbol{U} \supset \mathbf{Ker}(\boldsymbol{\Psi}_X)$ *and* $R(\boldsymbol{U}, Y)X = 0$.

Then the following inequality holds:

$$r \geq \dim M + \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y). \tag{2.2}$$

Proof. Let Z be an arbitrary element of $T_x(M)$. Then by the Gauss equation (2.1)

$$\begin{aligned} 0 &= -g\big(R(\boldsymbol{U}, Y)X, Z\big) \\ &= \big\langle \boldsymbol{\Psi}(\boldsymbol{U}, X), \boldsymbol{\Psi}(Y, Z)\big\rangle - \big\langle \boldsymbol{\Psi}(\boldsymbol{U}, Z), \boldsymbol{\Psi}(Y, X)\big\rangle \\ &= \big\langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z)\big\rangle - 0. \end{aligned}$$

Hence, we have $\big\langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z)\big\rangle = 0$. This implies that the image of $T_x(M)$ via the map $\boldsymbol{\Psi}_Y$ is included in the orthogonal complement of $\boldsymbol{\Psi}_X(\boldsymbol{U})$. Since $\dim \boldsymbol{\Psi}_X(\boldsymbol{U}) = \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$, we have $\dim \boldsymbol{\Psi}_Y(T_x(M)) \leq r - \dim \boldsymbol{U} + \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$. Moreover, since $\dim \boldsymbol{\Psi}_Y(T_x(M)) = \dim M - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, we immediately obtain the inequality (2.2). □

As is easily seen, the right side of the inequality (2.2) heavily depends on tangent vectors X, Y and $\boldsymbol{\Psi}$. Accordingly, only by (2.2) we cannot obtain an estimate for $\mathrm{Crank}(M)_x$. In the following sections, by applying Theorem 6 to the complex and quaternion projective spaces we will show Theorem 1.

3. Projective spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$

In this section we make several preparations that are needed in the succeeding sections. Hereafter, G/K denotes one of the following projective spaces:

(1) The complex projective spaces $P^n(\boldsymbol{C}) = SU(n+1)/S(U(n) \times U(1))$ $(n \geq 2)$.

(2) The quaternion projective spaces $P^n(\boldsymbol{H}) = Sp(n+1)/Sp(n) \times Sp(1)$ $(n \geq 2)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair (G, K). Let $\big(\ ,\ \big)$ be the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. We define a G-invariant Riemannian metric g of G/K by $g(X, Y) = \big(X, Y\big)$ $(X, Y \in \mathfrak{m})$, where we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. Since the curvature at o is given by $R(X, Y)Z = -\big[[X, Y], Z\big]$ $(X, Y, Z \in \mathfrak{m})$ (see Helgason [18]), the Gauss equation (2.1) in codimension r at o can be written as follows:

$$\big(\big[[X, Y], Z\big], W\big) = \big\langle \boldsymbol{\Psi}(X, Z), \boldsymbol{\Psi}(Y, W)\big\rangle - \big\langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Y, Z)\big\rangle, \tag{3.1}$$

where $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$, X, Y, Z and $W \in \mathfrak{m}$.

Let us take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then, since $\mathrm{rank}(G/K) = 1$, we have $\dim \mathfrak{a} = 1$. We call an element $\lambda \in \mathfrak{a}$ a *restricted root* when the subspaces $\mathfrak{k}(\lambda)$ $(\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)$ $(\subset \mathfrak{m})$ defined below are not non-trivial:

$$\begin{aligned} \mathfrak{k}(\lambda) &= \Big\{X \in \mathfrak{k} \,\big|\, \big[H, [H, X]\big] = -\big(\lambda, H\big)^2 X, \quad \forall H \in \mathfrak{a}\Big\}, \\ \mathfrak{m}(\lambda) &= \Big\{Y \in \mathfrak{m} \,\big|\, \big[H, [H, Y]\big] = -\big(\lambda, H\big)^2 Y, \quad \forall H \in \mathfrak{a}\Big\}. \end{aligned}$$

As is known, by use of a non-zero restricted root μ the set of non-zero restricted roots Σ can be written as $\Sigma = \{\pm\mu,\ \pm 2\mu\}$. Further, we have the following orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)},$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)},$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see § 5 of [7]).

For convenience, in the following we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)\,(|i| \leq 2)$ and $\mathfrak{k}_i = \mathfrak{m}_i = 0\,(|i| > 2)$ for any integer i. Then for $i, j = 0, 1, 2$ we have a formula:

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \ [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \ [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

We summarize in the following table the basic data for the spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ (see [18], [7]):

G/K	$\dim \mathfrak{m}_1 (= \dim \mathfrak{k}_1)$	$\dim \mathfrak{m}_2 (= \dim \mathfrak{k}_2)$
$P^n(\boldsymbol{C})(n \geq 2)$	$2(n-1)$	1
$P^n(\boldsymbol{H})(n \geq 2)$	$4(n-1)$	3

As is known, each non-zero element of $\mathfrak{m}$ is conjugate to a scalar multiple of μ under the action of the isotropy group $\mathrm{Ad}(K)$, because $\mathrm{rank}(P^n(\boldsymbol{C})) = \mathrm{rank}(P^n(\boldsymbol{H})) = 1$. More precisely we can show the following

Proposition 7. *Let $Y_i \in \mathfrak{m}_i\,(i = 0, 1, 2)$. Assume that $Y_i \neq 0$. Then there is an element $k_i \in K$ such that $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$.*

Proof. In the case $i = 0$ we have only to set $k_0 = e$, where e is the identity element of K.

Now assume $i = 1$ or 2. Set $X_i = [\mu, Y_i]$. Then we have $X_i \in \mathfrak{k}_i$. Further, we have $[X_i, [X_i, \mu]] \in \mathfrak{a}$, because $[X_i, [X_i, \mu]] \in \mathfrak{m}$ and $[\mu, [X_i, [X_i, \mu]]] = [[\mu, X_i], [X_i, \mu]] + [X_i, [\mu, [X_i, \mu]]] = 0$. Since

$$(\mu, [X_i, [X_i, \mu]]) = ([\mu, X_i], [X_i, \mu]) = ([\mu, [\mu, X_i]], X_i) = -i^2 (\mu, \mu)^2 (X_i, X_i),$$

we have $[X_i, [X_i, \mu]] = -i^2(\mu, \mu)(X_i, X_i)\mu$. By this equality and the fact $[X_i, \mu] = [[\mu, Y_i], \mu] = i^2(\mu, \mu)^2 Y_i$ we have

$$\mathrm{Ad}(\exp(tX_i))\mu = \cos(i|\mu||X_i|t)\mu + \frac{1}{i|\mu||X_i|}\sin(i|\mu||X_i|t)\,[X_i, \mu], \quad \forall t \in \boldsymbol{R}.$$

Define $t_i \in \boldsymbol{R}$ by $i|\mu||X_i|t_i = \pi/2$. Then, by setting $k_i = \exp(t_i X_i) \in K$, we easily get $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$. □

4. Pseudo-abelian subspaces

Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. We say that a subspace V of $\mathfrak{m}$ is *pseudo-abelian* if $[V, V] \subset \mathfrak{k}_0$. It is easily seen that a subspace V of $\mathfrak{m}$ is pseudo-abelian if and only if $[[V, V], \mu] = 0$, because $\mathrm{rank}(G/K) = 1$. We note that the pseudo-nullity $p(G/K)$ coincides with the the maximum dimension of pseudo-abelian subspaces in $\mathfrak{m}$ (see [4]). In [7] we have determined the pseudo-nullities for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$: $p(P^n(\boldsymbol{C})) = \max\{n-1, 2\}\,(n \geq 2)$; $p(P^n(\boldsymbol{H})) = \max\{n-1, 3\}\,(n \geq 2)$ (see Theorem 5.1 of [7]).

For later use, we here study more detailed facts about pseudo-abelian subspaces in $\mathfrak{m}$ for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$. We first prove

Lemma 8. *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V \cap \mathfrak{m}_i \neq 0$ for some $\mathfrak{m}_i$ $(i = 0, 1, 2)$, then $V \subset \mathfrak{m}_i$.*

Proof. Assume that $V \cap \mathfrak{m}_1 \neq 0$. Take a non-zero element $Y_1^0 \in V \cap \mathfrak{m}_1$. Let $Y = Y_0 + Y_1$ be an arbitrary element of V, where $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$; $Y_1 \in \mathfrak{m}_1$. Then we have $\left[Y_1^0, Y_0 + Y_1\right] = \left[Y_1^0, Y_0\right] + \left[Y_1^0, Y_1\right] \in \mathfrak{k}_0$. However, since $\left[Y_1^0, Y_0\right] \in \mathfrak{k}_1$ and $\left[Y_1^0, Y_1\right] \in \mathfrak{k}_0 + \mathfrak{k}_2$, we have $\left[Y_1^0, Y_0\right] = 0$. Therefore we have $Y_0 = 0$, because $\operatorname{rank}(G/K) = 1$. This proves $V \subset \mathfrak{m}_1$. The other cases $V \cap \mathfrak{a} \neq 0$ and $V \cap \mathfrak{m}_2 \neq 0$ are similarly dealt with. □

We say that a pseudo-abelian subspace V is *categorical* if it can be decomposed into a direct sum $V = V \cap \mathfrak{a} + V \cap \mathfrak{m}_1 + V \cap \mathfrak{m}_2$. By Lemma 8 we immediately have

Proposition 9. *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If V is categorical and $V \neq 0$, then V is contained in one of $\mathfrak{a}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$.*

By this proposition, we can easily estimate $\dim V$ for a categorical pseudo-abelian subspace V in $\mathfrak{m}$: $\dim V \leq 1$ if $V \subset \mathfrak{a}$; $\dim V \leq \dim \mathfrak{m}_2$ if $V \subset \mathfrak{m}_2$. In the case $V \subset \mathfrak{m}_1$ we proved in [7] $\dim V \leq n - 1$ (see Theorem 3.2 of [7]). For completeness, we review this proof and show an additional property of $V \subset \mathfrak{m}_1$.

Let $E(\mathfrak{m}_1)$ denote the space of all linear endomorphisms of $\mathfrak{m}_1$. Let $X \in \mathfrak{k}_2$. We define an element $X^\dagger \in E(\mathfrak{m}_1)$ by

$$X^\dagger(Y) = \left[X, Y\right], \quad Y \in \mathfrak{m}_1.$$

(Note that $\left[\mathfrak{k}_2, \mathfrak{m}_1\right] \subset \mathfrak{m}_1$.) It is easy to see that $X^\dagger$ is skew-symmetric with respect to the inner product $(\ ,\ )$. We denote by $\mathfrak{k}_2^\dagger$ the subspace of $E(\mathfrak{m}_1)$ consisting of all $X^\dagger$ $(X \in \mathfrak{k}_2)$. Set $\mathfrak{F}^\dagger = \boldsymbol{R}\mathbf{1}_{\mathfrak{m}_1} + \mathfrak{k}_2^\dagger$ $(\subset E(\mathfrak{m}_1))$, where $\mathbf{1}_{\mathfrak{m}_1}$ denotes the identity mapping of $\mathfrak{m}_1$. We have proved in [7] (Theorem 3.5) the following

Proposition 10. *Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. Then, $\mathfrak{F}^\dagger$ forms a subalgebra of $E(\mathfrak{m}_1)$, i.e., $\mathfrak{F}^\dagger$ is closed under addition and multiplication of $E(\mathfrak{m}_1)$. Further, in the case $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$, $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{C}$ of complex numbers and in the case $G/K = P^n(\boldsymbol{H})$ $(n \geq 2)$, $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{H}$ of quaternion numbers.*

We now set $f = \dim_{\boldsymbol{R}} \mathfrak{F}^\dagger$, i.e., $f = 2$ if $G/K = P^n(\boldsymbol{C})$; $f = 4$ if $G/K = P^n(\boldsymbol{H})$. By the definition we have $\dim \mathfrak{m}_2 = f - 1$, $\dim \mathfrak{m}_1 = (n-1)f$ and $\dim G/K = \dim \mathfrak{m} = nf$. As seen in Proposition 10, $\mathfrak{m}_1$ can be regarded as a vector space over the field $\mathfrak{F}^\dagger$. For an element $Y_1 \in \mathfrak{m}_1$ we denote by $\mathfrak{F}^\dagger(Y_1)$ the subspace of $\mathfrak{m}_1$ spanned by Y_1 over $\mathfrak{F}^\dagger$. Then we easily have $\mathfrak{F}^\dagger(\mathfrak{F}^\dagger(Y_1)) = \mathfrak{F}^\dagger(Y_1)$ and $\dim_{\boldsymbol{R}} \mathfrak{F}^\dagger(Y_1) = f$ if $Y_1 \neq 0$.

Lemma 11. *Let Y_1, $Y_1' \in \mathfrak{m}_1$. Then $\left[Y_1, Y_1'\right] \in \mathfrak{k}_0$ if and only if $\left(\mathfrak{k}_2^\dagger(Y_1), Y_1'\right) = 0$. Accordingly, a subspace $V \subset \mathfrak{m}_1$ is pseudo-abelian if and only if $\left(\mathfrak{k}_2^\dagger(V), V\right) = 0$.*

Proof. Since $[Y_1, Y_1'] \in \mathfrak{k}_0 + \mathfrak{k}_2$, $[Y_1, Y_1'] \in \mathfrak{k}_0$ holds if and only if $([Y_1, Y_1'], \mathfrak{k}_2) = 0$. Clearly, the last equality is equivalent to $(\mathfrak{k}_2^\dagger(Y_1), Y_1') = 0$. □

Utilizing the above lemma, we can show the following

Proposition 12. *Let V be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that $V \subset \mathfrak{m}_1$. Then:*

(1) $\dim \mathfrak{F}^\dagger(V) = f \dim V$. *Accordingly,* $\dim V \leq n-1$.

(2) *Let $\xi \in V$ $(\xi \neq 0)$. Then there is a subspace U of $\mathfrak{m}_1$ satisfying $U \supset V$, $[\xi, U] \subset \mathfrak{k}_0$ and* $\dim U = (n-2)f + 1$.

Proof. Let $\{Y_1^1, \dots, Y_1^s\}$ $(s = \dim V)$ be an orthonormal basis of V. Let i, j be integers such that $1 \leq i \neq j \leq s$. Then, since $(\mathfrak{k}_2^\dagger(Y_1^i), Y_1^j) = (Y_1^i, \mathfrak{k}_2^\dagger(Y_1^j)) = 0$ (see Lemma 11) and since $(\mathfrak{k}_2^\dagger)^2 \subset \mathfrak{F}^\dagger$, we have

$$(\mathfrak{F}^\dagger(Y_1^i), \mathfrak{F}^\dagger(Y_1^j)) = (\boldsymbol{R}Y_1^i + \mathfrak{k}_2^\dagger(Y_1^i), \boldsymbol{R}Y_1^j + \mathfrak{k}_2^\dagger(Y_1^j)) \subset (Y_1^i, (\mathfrak{k}_2^\dagger)^2(Y_1^j)) = 0.$$

This proves $\mathfrak{F}^\dagger(V) = \sum_{1 \leq i \leq s} \mathfrak{F}^\dagger(Y_1^i)$ (orthogonal direct sum) and hence $\dim_{\boldsymbol{R}} \mathfrak{F}^\dagger(V) = sf$. Therefore we have $s \leq n-1$, because $\dim \mathfrak{m}_1 = (n-1)f$.

Next we prove (2). Since V is pseudo-abelian and $\xi \in V$, we have $(\mathfrak{k}_2^\dagger(\xi), V) = 0$. Let U be the orthogonal complement of $\mathfrak{k}_2^\dagger(\xi)$ in $\mathfrak{m}_1$. Then U satisfies $U \supset V$ and $[\xi, U] \subset \mathfrak{k}_0$ (see Lemma 11). Moreover, since $\dim \mathfrak{k}_2^\dagger(\xi) = f - 1$ and $\dim \mathfrak{m}_1 = (n-1)f$, we immediately obtain the equality $\dim U = (n-2)f + 1$. □

Finally, we refer to non-categorical pseudo-abelian subspaces. Let V be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that V is not categorical, i.e., V cannot be represented by a direct sum of subspaces $V \cap \mathfrak{a}$, $V \cap \mathfrak{m}_1$ and $V \cap \mathfrak{m}_2$. Then it is clear that $V \not\subset \mathfrak{a}$, $V \not\subset \mathfrak{m}_1$ and $V \not\subset \mathfrak{m}_2$. In view of Lemma 8, we know that $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$. Apparently, this condition is sufficient for a pseudo-abelian subspace V to be non-categorical. Hence we have

Proposition 13. *Let V be a pseudo-abelian subspace of $\mathfrak{m}$ such that $V \neq 0$.*

(1) *V is non-categorical if and only if* $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$.

(2) *If V is non-categorical, then* $\dim V \leq 2$.

For the proof of (2), see Proposition 5.2 (1) of [7].

5. Proof of Theorem 1

Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. In the following we assume that the Gauss equation in codimension r admits a solution $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$. We first prove

Lemma 14. *Let $X \in \mathfrak{m}$ $(X \neq 0)$ and let k be an element of K satisfying* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. *Then* $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_X)$ *is a pseudo-abelian subspace of* $\mathfrak{m}$.

Proof. By Lemma 3 we have $\big[[\mathbf{Ker}(\Psi_X), \mathbf{Ker}(\Psi_X)], X\big] = 0$. Applying $\mathrm{Ad}(k^{-1})$ to this equality, we have $\big[[\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X), \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X)], \mu\big] = 0$. This proves that $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. □

Let $X \in \mathfrak{m}\,(X \neq 0)$. If $\mathbf{Ker}(\Psi_X) = 0$, then we say X is *of type* P_{inj}. Now assume $\mathbf{Ker}(\Psi_X) \neq 0$. Let $k \in K$ be an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. As is shown in Lemma 14, $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. If $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X)$ is categorical and is contained in $\mathfrak{m}_i\,(i = 0, 1, 2)$, then we say X is *of type* $P_i\,(i = 0, 1, 2)$. We also say X is *of type* P_{non} if $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X)$ is non-categorical, i.e., $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X) \cap \mathfrak{m}_i = 0\,(i = 0, 1, 2)$.

The following lemma asserts that the type of X does not depend on the choice of $k \in K$ satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$.

Lemma 15. *Let $X \in \mathfrak{m}\,(X \neq 0)$. Let $i = 0$, 1 or 2 and let $k_j\,(j = 1, 2)$ be elements of K satisfying $\mathrm{Ad}(k_j)\mu \in \boldsymbol{R}X$. Then:*

(1) $\mathrm{Ad}(k_1^{-1})\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$ *if and only if* $\mathrm{Ad}(k_2^{-1})\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$.
(2) $\mathrm{Ad}(k_1^{-1})\mathbf{Ker}(\Psi_X) \cap \mathfrak{m}_i = 0$ *if and only if* $\mathrm{Ad}(k_2^{-1})\mathbf{Ker}(\Psi_X) \cap \mathfrak{m}_i = 0$.

Proof. Set $k' = k_1^{-1}k_2 \in K$. By the assumption we have $\mathrm{Ad}(k')\mu = \pm\mu$. Therefore it is easily seen that $\mathrm{Ad}(k')\mathfrak{m}_i = \mathfrak{m}_i$ for any $i = 0, 1, 2$. Since $\mathrm{Ad}(k')\mathrm{Ad}(k_2^{-1}) = \mathrm{Ad}(k_1^{-1})$, the lemma follows immediately. □

Let us denote by $\mathfrak{p}_i\,(i = 0, 1, 2, non, inj)$ the subset of $\mathfrak{m}$ consisting of all elements of type P_i. Then it is clear that

$$\mathfrak{m}\backslash\{0\} = \mathfrak{p}_0 \cup \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_{non} \cup \mathfrak{p}_{inj} \quad \text{(disjoint union)}. \tag{5.1}$$

Proposition 16. *Let $X, Y \in \mathfrak{m}\,(X \neq 0, Y \neq 0)$. Assume that $\Psi(X, Y) = 0$. Then $X \in \mathfrak{p}_i$ if and only if $Y \in \mathfrak{p}_i\,(i = 0, 1, 2, non)$.*

Proof. We note that under the assumption $\Psi(X, Y) = 0$ we have $X \notin \mathfrak{p}_{inj}$ and $Y \notin \mathfrak{p}_{inj}$, because $Y \in \mathbf{Ker}(\Psi_X)$ and $X \in \mathbf{Ker}(\Psi_Y)$.

First consider the case $X \in \mathfrak{p}_i\,(i = 0, 1, 2)$. Let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})Y \in \mathfrak{m}_i$, because $\mathrm{Ad}(k^{-1})Y \in \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$. Take an element $k' \in K$ satisfying $\mathrm{Ad}(k'^{\pm 1})\mu \in \boldsymbol{R}\mathrm{Ad}(k^{-1})Y$ and set $k'' = kk'$ (see Proposition 7). Then we have $\mathrm{Ad}(k'')\mu = \mathrm{Ad}(k)\mathrm{Ad}(k')\mu \in \mathrm{Ad}(k)\boldsymbol{R}\mathrm{Ad}(k^{-1})Y = \boldsymbol{R}Y$ and $\mathrm{Ad}(k''^{-1})X = \mathrm{Ad}(k'^{-1})\mathrm{Ad}(k^{-1})X \in \boldsymbol{R}\mathrm{Ad}(k'^{-1})\mu = \boldsymbol{R}\mathrm{Ad}(k^{-1})Y \subset \mathfrak{m}_i$. Since $X \in \mathbf{Ker}(\Psi_Y)$, it follows that $\mathrm{Ad}(k''^{-1})\mathbf{Ker}(\Psi_Y) \cap \mathfrak{m}_i \neq 0$. Hence $\mathrm{Ad}(k''^{-1})\mathbf{Ker}(\Psi_Y)$ is categorical (see Proposition 13) and $\mathrm{Ad}(k''^{-1})\mathbf{Ker}(\Psi_Y) \subset \mathfrak{m}_i$ (see Proposition 9). This means $Y \in \mathfrak{p}_i$. The converse can be proved in the same manner.

By these arguments we know that $X \in \mathfrak{p}_{non}$ if and only if $Y \in \mathfrak{p}_{non}$. □

Lemma 17. *Let $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$ or $P^n(\boldsymbol{H})\,(n \geq 2)$. Then:*

(1) $\mathfrak{p}_0 = \emptyset$.

(2) *Let* $X \in \mathfrak{m}\,(X \neq 0)$. *Then*:

$$\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \leq \begin{cases} n-1, & \text{if } X \in \mathfrak{p}_1; \\ f-1, & \text{if } X \in \mathfrak{p}_2; \\ 2, & \text{if } X \in \mathfrak{p}_{non}. \end{cases} \tag{5.2}$$

Proof. Suppose that $\mathfrak{p}_0 \neq \emptyset$. Let $X \in \mathfrak{p}_0$ and let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_X) \subset \mathfrak{a} = \boldsymbol{R}\mu$. Hence we have $\mathbf{Ker}(\boldsymbol{\Psi}_X) = \boldsymbol{R}\mathrm{Ad}(k)\mu = \boldsymbol{R}X$, i.e., $\boldsymbol{\Psi}(X,X) = 0$. Let $Y \in \mathfrak{m}$ such that $Y \notin \boldsymbol{R}X$. By (3.1) we have

$$\big(\big[[X,Y],X\big],Y\big) = \big\langle \boldsymbol{\Psi}(X,X), \boldsymbol{\Psi}(Y,Y)\big\rangle - \big\langle \boldsymbol{\Psi}(X,Y), \boldsymbol{\Psi}(Y,X)\big\rangle = -\big\langle \boldsymbol{\Psi}_X(Y), \boldsymbol{\Psi}_X(Y)\big\rangle.$$

Since G/K is of positive curvature, the left side of the above equality is ≥ 0. Therefore we have $\boldsymbol{\Psi}_X(Y) = 0$, which contradicts $Y \notin \boldsymbol{R}X$. Thus we have, $\mathfrak{p}_0 = \emptyset$.

The assertion (2) follows from Propositions 12, Proposition 13, $\dim \mathfrak{m}_2 = f-1$ and the discussions in the previous section. □

Proposition 18. *Let* $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$ *or* $P^n(\boldsymbol{H})\,(n \geq 2)$. *Then*:

(1) $\mathfrak{p}_{inj} = \emptyset$ *if* $r \leq nf-1$;
(2) $\mathfrak{p}_1 = \emptyset$ *if* $r \leq 2(n-1)(f-1)$;
(3) $\mathfrak{p}_2 = \emptyset$ *if* $r \leq (n-1)f$;
(4) $\mathfrak{p}_{non} = \emptyset$ *if* $r \leq nf-3$.

Proof. We first note that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq \dim G/K - r = nf - r$ holds for any $X \in \mathfrak{m}$. By this fact we can easily prove (1), (3) and (4). In fact, if $r \leq nf-1$, then it is clear that $\mathbf{Ker}(\boldsymbol{\Psi}_X) \neq 0$ for any $X \in \mathfrak{m}$. Hence $X \notin \mathfrak{p}_{inj}$, which implies $\mathfrak{p}_{inj} = \emptyset$. Similarly, if $r \leq (n-1)f$ (resp. $r \leq nf-3$), then $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq f$ (resp. $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq 3$) holds for any $X \in \mathfrak{m}$ and hence $\mathfrak{p}_2 = \emptyset$ (resp. $\mathfrak{p}_{non} = \emptyset$) (see Lemma 17).

Next we prove (2). Suppose that $\mathfrak{p}_1 \neq \emptyset$. Let $X \in \mathfrak{p}_1$. Take $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$ and set $V = \mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_X)$. Then V is a pseudo-abelian subspace such that $V \subset \mathfrak{m}_1$. Consequently, by Lemma 17 we have $\dim V \leq n-1$.

Now let us take a non-zero element $\xi \in V$ and a subspace $U \subset \mathfrak{m}_1$ satisfying $U \supset V$, $[\xi, U] \subset \mathfrak{k}_0$ and $\dim U = (n-2)f+1$ (see Proposition 12 (2)). Put $Y = \mathrm{Ad}(k)\xi\,(\in \mathbf{Ker}(\boldsymbol{\Psi}_X))$ and $\boldsymbol{U} = \mathrm{Ad}(k)U\,(\subset \mathfrak{m})$. Then we have $\boldsymbol{\Psi}(X,Y) = 0$ and $\boldsymbol{U} \supset \mathbf{Ker}(\boldsymbol{\Psi}_X)$. Moreover, we have $\big[[\boldsymbol{U},Y],X\big] = 0$, because $\big[[\boldsymbol{U},Y],X\big] = \mathrm{Ad}(k)\big[[U,\xi],\mu\big] = 0$. Therefore, by Theorem 6 we have the following inequality:

$$r \geq nf + (n-2)f + 1 - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y).$$

Since X and $Y \in \mathfrak{p}_1$ (see Proposition 16), it follows that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \leq n-1$ and $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \leq n-1$ (see Lemma 17). Consequently, we have $r \geq 2(n-1)(f-1)+1$, which proves (2). □

We are now in a position to prove Theorem 1. If there is a solution $\boldsymbol{\Psi}$ of the Gauss equation in codimension r, then at least one of the sets $\mathfrak{p}_{inj}$, $\mathfrak{p}_0$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_{non}$ is not empty (see (5.1)). Therefore, in view of Lemma 17 (1) and Proposition 18, we have $r \geq 1 + \min\{nf-1, 2(n-1)(f-1), (n-1)f, nf-3\}$. Accordingly, we have $r \geq 2n-2$ if $G/K = P^n(\boldsymbol{C})$ and $r \geq 4n-3$ if $G/K = P^n(\boldsymbol{H})$. Hence, $\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-2$ and $\mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-3$. This, together with Lemma 2, shows Theorem 1. □

Remark 1. The proof of Theorem 1 stated above is effective in the case $n = 2$. We thereby have $\mathrm{Crank}(P^2(\boldsymbol{C})) \geq 2$ and $\mathrm{Crank}(P^2(\boldsymbol{H})) \geq 5$. However, for the spaces $P^2(\boldsymbol{C})$ and $P^2(\boldsymbol{H})$, we have already known the best results: $\mathrm{Crank}(P^2(\boldsymbol{C})) = 3$ (see [1]) and $\mathrm{class}(P^2(\boldsymbol{H})) = \mathrm{Crank}(P^2(\boldsymbol{H})) = 6$ (see [8]).

As for the class number of $P^2(\boldsymbol{C})$ we have $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or 4 (see Lemma 2 and Introduction). It is still an open question whether $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or not (cf. [20]).

Remark 2. Consider the following two cases:

(1) $G/K = P^n(\boldsymbol{C})\,(n \geq 3)$ and $r = 2n-2$;
(2) $G/K = P^n(\boldsymbol{H})\,(n \geq 3)$ and $r = 4n-3$.

If there is a solution $\boldsymbol{\Psi}$ of the Gauss equation in codimension r, then it is shown by Lemma 17 (1) and Proposition 18 that $\boldsymbol{\Psi}$ must satisfies the following condition:

Case (1) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m}\backslash\{0\} = \mathfrak{p}_{non}$;
Case (2) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_{non} = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m}\backslash\{0\} = \mathfrak{p}_2$.

We conjecture that in both cases (1) and (2) there are no such solutions $\boldsymbol{\Psi}$. In other words:

$$\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-1\,(n \geq 3); \quad \mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-2\,(n \geq 3).$$

If this is true, then we obtain an improvement of Theorem 1:

$$\mathrm{class}(P^n(\boldsymbol{C})) \geq 2n-1\,(n \geq 3); \quad \mathrm{class}(P^n(\boldsymbol{H})) \geq 4n-2\,(n \geq 3).$$

References

[1] Y. Agaoka, *On the curvature of Riemannian submanifolds of codimension* 2, Hokkaido Math. J. **14** (1985), 107–135.

[2] ———, *A note on local isometric imbeddings of complex projective spaces*, J. Math. Kyoto Univ. **27** (1987), 501–505.

[3] Y. Agaoka and E. Kaneda, *On local isometric immersions of Riemannian symmetric spaces*, Tôhoku Math. J. **36** (1984), 107–140.

[4] ———, *An estimate on the codimension of local isometric imbeddings of compact Lie groups*, Hiroshima Math. J. **24** (1994), 77–110.

[5] ———, *Local isometric imbeddings of symplectic groups*, Geometriae Dedicata **71** (1998), 75–82.

[6] ———, *Strongly orthogonal subsets in root systems*, Hokkaido Math. J. **31** (2002), 107–136.

[7] ———, *A lower bound for the curvature invariant $p(G/K)$ associated with a Riemannian symmetric space G/K*, Hokkaido Math. J. **33** (2004), 153–184.

[8] ———, *Local isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$*, Hokkaido Math. J. **33** (2004), 399–412.

[9] ———, *Rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(\boldsymbol{Cay})$*, (*to appear in Hokkaido Math. J.*).

[10] ———, *Rigidity of the canonical isometric imbedding of the quaternion projective plane* $P^2(\boldsymbol{H})$, (*to appear in Hokkaido Math. J.*).

[11] ———, *Local isometric imbeddings of Grassmann manifolds*, (*in preparation*).

[12] E. J. Berger, *The Gauss map and isometric embedding*, Ph. D. Thesis, Harvard Univ. (1981).

[13] E. Berger, R. Bryant and P. Griffiths, *The Gauss equations and rigidity of isometric embeddings*, Duke Math. J. **50** (1983), 803–892.

[14] E. Cartan, *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Polon. Math. **6** (1927), 1–7.

[15] R. E. Greene, Isometric Embeddings of Riemannian and Pseudo-Riemannian Manifolds, Mem. Amer. Math. Soc. **97** (1970).

[16] R. E. Greene and H. Jacobowitz, *Analytic isometric embeddings*, Ann. of Math. **93** (1971), 189–204.

[17] M. L. Gromov and V. A. Rokhlin, *Embeddings and immersions in Riemannian geometry*, Russian Math. Surveys **25**(5) (1970), 1–57.

[18] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York (1978).

[19] M. Janet, *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Polon. Math. **5** (1926), 38–43.

[20] E. Kaneda, *On the Gauss-Codazzi equations*, Hokkaido Math. J. **19** (1990), 189–213.

[21] E. Kaneda and N. Tanaka, *Rigidity for isometric imbeddings*, J. Math. Kyoto Univ. **18** (1978), 1–70.

[22] S. Kobayashi, *Isometric imbeddings of compact symmetric spaces*, Tôhoku Math. J. **20** (1968), 21–25.

[23] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry II, Wiley-Interscience, New York (1969).

[24] J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.

(Yoshio AGAOKA)
Faculty of Integrated Arts and Sciences, Hiroshima University
1-7-1 Kagamiyama, Higashi-Hiroshima City, Hiroshima, 739-8521, Japan
E-mail address: agaoka@mis.hiroshima-u.ac.jp

(Eiji KANEDA)
Faculty of Foreign Studies, Osaka University of Foreign Studies
8-1-1 Aomadani-Higashi, Minoo City, Osaka, 562-8558, Japan
E-mail address: kaneda@osaka-gaidai.ac.jp