# RIGIDITY OF THE CANONICAL ISOMETRIC IMBEDDING OF THE SYMPLECTIC GROUP $Sp(n)$

YOSHIO AGAOKA AND EIJI KANEDA

Abstract. In this paper, we discuss the rigidity of $Sp(n)$ as a Riemannian submanifold of $M(n,n;\mathbb{H})$. We prove that the inclusion map $\boldsymbol{f}_0$, which is called the canonical isometric imbedding of $Sp(n)$, is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set $U(\subset Sp(n))$ into $\boldsymbol{R}^{4n^2}$ ($\cong M(n,n;\mathbb{H})$) coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{4n^2}$, i.e., there is a euclidean transformation $a$ of $\boldsymbol{R}^{4n^2}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on $U$.

## Introduction

The subject of this paper is to prove the rigidity of the symplectic group $Sp(n)$ as a Riemannian submanifold of the space of matrices over the field of quaternion numbers.

Let $M(n,n;\mathbb{H})$ be the space of $n \times n$-matrices over the field $\mathbb{H}$ of quaternion numbers. Considering $M(n,n;\mathbb{H})$ as a real vector space, we define a bilinear form $\nu$ on $M(n,n;\mathbb{H})$ by setting

$$\nu(X,Y) = \mathrm{Re}(\mathrm{Trace}({}^t\bar{X}\,Y)), \quad X, Y \in M(n,n;\mathbb{H}).$$

It is easily seen that $\nu$ defines an inner product on $M(n,n;\mathbb{H})$. With this inner product $\nu$ we can regard $M(n,n;\mathbb{H})$ as the euclidean space $\mathbb{R}^{4n^2}$. The *symplectic group* $Sp(n)$ is given by a submanifold of $M(n,n;\mathbb{H})$ consisting of all matrices $g \in M(n,n;\mathbb{H})$ satisfying $g\,{}^t\bar{g} = {}^t\bar{g}g = I_n$, where $I_n$ is the identity matrix of degree $n$. The induced metric on $Sp(n)$, which is denoted by the same symbol $\nu$, is bi-invariant on $Sp(n)$. The inclusion map $\boldsymbol{f}_0\colon Sp(n) \longrightarrow M(n,n;\mathbb{H}) \cong \mathbb{R}^{4n^2}$ gives an isometric imbedding of the Riemannian manifold $(Sp(n),\nu)$ into $\mathbb{R}^{4n^2}$ and is called the *canonical isometric imbedding* of $Sp(n)$ into $\mathbb{R}^{4n^2}$ (cf. Kobayashi [17]). In this paper we will discuss the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$.

Let $M$ be a Riemannian manifold and let $\boldsymbol{f}$ be an isometric imbedding of $M$ into the euclidean space $\mathbb{R}^N$. By definition $\boldsymbol{f}$ is called *strongly rigid* when $\boldsymbol{f}$ is rigid even if we restrict $\boldsymbol{f}$ to any connected open set of $M$, i.e., for any isometric immersion $\boldsymbol{f}'$ of a connected open set $U\,(\subset M)$ into $\mathbb{R}^N$ there exists a euclidean transformation $a$ of $\mathbb{R}^N$ satisfying $\boldsymbol{f}' = a\boldsymbol{f}$ on $U$. In [8] and [9] we showed that the canonical isometric imbeddings of the quaternion projective plane $P^2(\mathbb{H})$ and the Cayley projective plane $P^2(\mathbb{CAY})$ are strongly rigid.

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Concerning the canonical isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)$ into $\mathbb{R}^{4n^2}$, the following results are known:

(1) In the case where $n = 1$, $\boldsymbol{f}_0$ is just the standard isometric imbedding of $S^3$ ($\cong Sp(1)$) into $\mathbb{R}^4$ with radius 1, which is a typical example of isometric imbeddings with type number 3. Accordingly, by Allendoefer [12] $\boldsymbol{f}_0$ is known to be strongly rigid.

(2) By investigating the Gauss equation of $Sp(2)$ in codimension 6 (for the definition, see §2 below), Agaoka [1] showed that the set of solutions of the Gauss equation is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of $\boldsymbol{f}_0$. Utilizing this fact, Agaoka proved that $\boldsymbol{f}_0$ is strongly rigid when $n = 2$.

(3) Kaneda [15] proved that $\boldsymbol{f}_0$ ($n \geq 1$) is globally rigid in the sense of Tanaka [19], i.e., if two differentiable maps $\boldsymbol{f}_i$ ($i = 1, 2$) of $Sp(n)$ into $\mathbb{R}^{4n^2}$ lie both near to $\boldsymbol{f}_0$ with respect to $C^3$-topology, and if they induce the same Riemannian metric on $Sp(n)$, then there is a euclidean transformation $a$ of $\mathbb{R}^{4n^2}$ such that $\boldsymbol{f}_2 = a\boldsymbol{f}_1$.

(4) By determining the pseudo-nullity of $Sp(n)$ ($n \geq 1$), Agaoka-Kaneda [4] proved that $\mathbb{R}^{4n^2}$ is the least dimensional euclidean space into which $Sp(n)$ can be locally isometrically immersed. (For the definition of the pseudo-nullity, see §1.) In other words, $Sp(n)$ ($n \geq 1$) cannot be isometrically immersed into $\mathbb{R}^{4n^2-1}$ even locally.

In this paper, we will extend these results (1) $\sim$ (4) in the following strongest sense:

**Theorem 1.** *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of the symplectic group $Sp(n)$ into the euclidean space $\mathbb{R}^{4n^2}$. Then $\boldsymbol{f}_0$ is strongly rigid, i.e., for any isometric immersion $\boldsymbol{f}$ of a connected open set $U$ ($\subset Sp(n)$) into $\mathbb{R}^{4n^2}$ there is a euclidean transformation $a$ of $\mathbb{R}^{4n^2}$ satisfying $\boldsymbol{f} = a\boldsymbol{f}_0$ on $U$.*

It should be noted that $Sp(n)$ ($n \geq 1$) are the first example that the canonical isometric imbeddings of a series of Riemannian symmetric spaces parametrized by the rank are strongly rigid. The method of our proof is quite similar to the methods adopted in [8] and [9]. We first make a preparatory study on pseudo-abelian subspaces of $\mathfrak{sp}(n)$, which is the Lie algebra of $Sp(n)$. Utilizing the knowledge about the pseudo-abelian subspaces of maximum dimension, we determine the set of all solutions of the Gauss equation of $Sp(n)$ in codimension $2n^2 - n$ ($= 4n^2 - \dim Sp(n)$). Under this situation, it will be shown that the set of solutions is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of $\boldsymbol{f}_0$. Therefore by the theorem of coincidence (Theorem 5 of [8, pp.335–336]) we can establish our rigidity theorem of $Sp(n)$ (Theorem 1).

Throughout this paper we will assume the differentiability of class $C^\infty$. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [14]. For the quaternion numbers and the symplectic group $Sp(n)$, see Chevalley [13].

## 1. The pseudo-nullity of $Sp(n)$

In this section we study the pseudo-nullity of $Sp(n)$. We first recall the notion of a pseudo-abelian subspace (precisely, see [3]). Let $G$ be a compact simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A subspace $W \subset \mathfrak{g}$ is called *pseudo-abelian with respect to* $\mathfrak{h}$ (or simply, *pseudo-abelian*) if it satisfies $[W, W] \subset \mathfrak{h}$. The maximum dimension of pseudo-abelian subspaces, which does not depend on the choice of a Cartan subalgebra $\mathfrak{h}$, is called the *pseudo-nullity* of $G$ and is denoted by $p_G$. The pseudo-nullity of the symplectic group $Sp(n)$ has been already determined:

**Theorem 2** (see [4]). *For the symplectic group $G = Sp(n)$ $(n \geq 1)$, the pseudo-nullity is equal to $2n$, i.e., $p_{Sp(n)} = 2n$.*

In what follows we determine the pseudo-abelian subspace $W$ of $\mathfrak{sp}(n)$ which attains the maximum dimension, i.e., $\dim W = p_{Sp(n)} = 2n$. First recall the field of quaternion numbers: Let $\mathbb{R}$ be the field of real numbers. The field $\mathbb{H}$ of quaternion numbers is an algebra over $\mathbb{R}$ generated by the elements $e^0$, $e^1$, $e^2$ and $e^3$ satisfying

(1) $e^0 e^i = e^i e^0 = e^i$ $(i = 0, 1, 2, 3)$;
(2) $(e^i)^2 = -e^0$ $(i = 1, 2, 3)$;
(3) For each permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ it holds $e^i e^j = \varepsilon(ijk) e^k$, where $\varepsilon(ijk) = 1$ (resp. $\varepsilon(ijk) = -1$) if $\{i, j, k\}$ is an even (resp. odd) permutation.

From (1) we can see that $e^0$ is a unit element of $\mathbb{H}$. Let us simply express the element $ae^0$ $(a \in \mathbb{R})$ as $a$. In this meaning $\mathbb{R}$ is contained in $\mathbb{H}$ and forms a subfield of $\mathbb{H}$.

Let $f \in \mathbb{H}$. Then $f$ may be written in the form $f = f_0 + \sum_{i=1}^3 f_i e^i$, where $f_0, f_1, f_2, f_3 \in \mathbb{R}$. As usual we define the real part and the conjugate of $f$ as follows: $\mathrm{Re}(f) = f_0$; $\bar{f} = f_0 - \sum_{i=1}^3 f_i e^i$. Then we have $\mathrm{Re}(f) = \mathrm{Re}(\bar{f})$, $f\bar{f} = \bar{f}f = \sum_{i=0}^3 f_i^2$. Moreover:

$$\mathrm{Re}(fh) = \mathrm{Re}(hf), \quad \overline{fh} = \bar{h}\bar{f}, \qquad f, h \in \mathbb{H}.$$

Let $i = 1$, 2 or 3. Define a subset $\mathbb{C}^i$ of $\mathbb{H}$ by $\mathbb{C}^i = \mathbb{R} + \mathbb{R}e^i$. It is easily seen that $\mathbb{C}^i$ forms a subfield of $\mathbb{H}$ and is isomorphic to the field $\mathbb{C}$ of complex numbers. We also define a subset $\mathbb{D}^i$ of $\mathbb{H}$ by $\mathbb{D}^i = \mathbb{R}e^j + \mathbb{R}e^k$, where $j$ and $k$ are so chosen that $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$. Then it is clear that

$$\mathbb{C}^i \mathbb{D}^i = \mathbb{D}^i \mathbb{C}^i = \mathbb{D}^i; \qquad \mathbb{D}^i \mathbb{D}^i = \mathbb{C}^i.$$

In the following we denote by $M(p, q; \mathbb{H})$ the space of $p \times q$-matrices over $\mathbb{H}$. As stated in Introduction, the symplectic group $Sp(n)$ is considered as a submanifold of $M(n, n; \mathbb{H}) \cong \mathbb{R}^{4n^2}$. As usual, we identify the tangent space of $Sp(n)$ at the identity $I_n \in Sp(n)$ with the Lie algebra $\mathfrak{sp}(n)$, which is consisting of all matrices $X \in M(n, n; \mathbb{H})$ satisfying ${}^t\bar{X} = -X$. Let us denote by $E_{st}$ $(1 \leq s, t \leq n)$ the matrix of $M(n, n; \mathbb{H})$ such that the $(s, t)$-component is 1 and the others are 0. We define subspaces $\mathfrak{h}(n)^i$ and $\mathfrak{p}(n)^i$

of $\mathfrak{sp}(n)$ by

$$\mathfrak{h}(n)^i = \sum_{s=1}^{n} \mathbb{R}e^i E_{ss}; \qquad \mathfrak{p}(n)^i = \sum_{s=1}^{n} \mathbb{D}^i E_{ss}.$$

As is well-known, $\mathfrak{h}(n)^i$ is a Cartan subalgebra of $\mathfrak{sp}(n)$. Moreover:

**Proposition 3.** *Let $i = 1$, $2$ or $3$. Then, $\mathfrak{p}(n)^i$ is pseudo-abelian with respect to $\mathfrak{h}(n)^i$ with* $\dim \mathfrak{p}(n)^i = p_{Sp(n)}$.

**Proof.** It is clear that $\dim \mathfrak{p}(n)^i = 2n$. Let $X = \sum_s u_s E_{ss}$, $Y = \sum_s v_s E_{ss} \in \mathfrak{p}(n)^i$, where $u_s$, $v_s \in \mathbb{D}^i$. Then, since $E_{ss}E_{ss} = E_{ss}$ and $E_{ss}E_{s's'} = 0\,(s \neq s')$, we have $[X, Y] = \sum_s (u_s v_s - v_s u_s) E_{ss}$. Since $u_s$, $v_s \in \mathbb{D}^i$, it follows that $u_s v_s$, $v_s u_s \in \mathbb{C}^i$ and $u_s v_s - v_s u_s \in \mathbb{R}e^i$. Hence $[X, Y] \in \mathfrak{h}(n)^i$, proving $[\mathfrak{p}(n)^i, \mathfrak{p}(n)^i] \subset \mathfrak{h}(n)^i$. □

Further, the space $\mathfrak{p}(n)^i$ is the only pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ of dimension $p_{Sp(n)}$. In fact, we have

**Theorem 4.** *Let $i = 1$, $2$ or $3$. Let $W$ be a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying* $\dim W = p_{Sp(n)}$. *Then $W = \mathfrak{p}(n)^i$.*

In the rest of this section we prove this theorem. Let $X = \sum_{st} \xi_{st} E_{st} \in M(n, n; \mathbb{H})$. We denote by $x_p = (\xi_{p1}, \cdots, \xi_{pn}) \in M(1, n; \mathbb{H})$ the $p$-th row of $X$ and by $x^q = {}^t(\xi_{1q}, \cdots, \xi_{nq}) \in M(n, 1; \mathbb{H})$ the $q$-th column of $X$. Then we may write

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \big(x^1, \ldots, x^n\big).$$

As is easily seen, $X \in \mathfrak{sp}(n)$ if and only if

$${}^t\bar{x}_p + x^p = 0 \quad (1 \leq p \leq n). \tag{1.1}$$

Let $X = \big(x^1, \ldots, x^n\big)$, $Y = \big(y^1, \ldots, y^n\big) \in \mathfrak{sp}(n)$. Then $[X, Y] \in \mathfrak{h}(n)^i$ if and only if the following conditions are satisfied:

$$\big(x^p, y^q\big) = \big(y^p, x^q\big) \quad (1 \leq p < q \leq n), \tag{1.2}$$

$$\big(x^r, y^r\big) \in \mathbb{C}^i \quad (1 \leq r \leq n), \tag{1.3}$$

where $\big(\, ,\, \big)$ denotes the inner product of $M(n, 1; \mathbb{H})$ defined by $\big(\xi, \eta\big) = {}^t\bar{\xi}\eta$ for $\xi$, $\eta \in M(n, 1; \mathbb{H})$. Then we note the following formula:

$$\overline{\big(\xi, \eta\big)} = \big(\eta, \xi\big), \quad \big(\xi f, \eta\big) = \bar{f}\big(\xi, \eta\big), \quad \big(\xi, \eta f\big) = \big(\xi, \eta\big) f, \quad f \in \mathbb{H}. \tag{1.4}$$

Now we start the proof of Theorem 4 by induction on $n$. First consider the case $n = 1$. In a natural way we identify $M(1, 1; \mathbb{H})$ with $\mathbb{H}$. Then by (1.1) we know that $w = a_0 + \sum_{j=1}^{3} a_j e^j \in \mathbb{H}$ belongs to $\mathfrak{sp}(1)$ if and only if $a_0 = 0$. Let $W$ be a pseudo-abelian subspace of $\mathfrak{sp}(1)$ with respect to $\mathfrak{h}(1)^i$ with $\dim W = 2$. Suppose that $W \neq \mathbb{D}^i$. Take a basis $\{w, w'\}$ of $W$ such that $w \notin \mathbb{D}^i$, i.e., $w$ is an element written in the form

$w = \sum_{j=1}^{3} a_j e^j$, where $a_i \neq 0$. By subtracting a scalar multiple of $w$ from $w'$ if necessary, we may assume that $w' \in \mathbb{D}^i$. Then we have $ww' = (\sum_{j\neq i} a_j e^j)w' + a_i e^i w'$, $(\sum_{j\neq i} a_j e^j)w' \in \mathbb{C}^i$ and $a_i e^i w' \in \mathbb{D}^i$. On the other hand, by (1.3) we have $ww' = -\bar{w}w' \in \mathbb{C}^i$. This is impossible because $a_i e^i w' \neq 0$. Hence we have $W = \mathbb{D}^i = \mathfrak{p}(1)^i$, showing that Theorem 4 is true when $n = 1$.

We now assume that $n \geq 2$ and Theorem 4 is true for any $n' \, (1 \leq n' < n)$. For simplicity, we regard $\mathfrak{sp}(s) \, (1 \leq s < n)$ as a subalgebra of $\mathfrak{sp}(n)$ in the following manner:

$$\mathfrak{sp}(s) \ni X \longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sp}(n).$$

Let $W$ be a pseudo-abelian subspace of $\mathfrak{sp}(n)$ with respect to $\mathfrak{h}(n)^i$. As in [4] we define an ascending chain of subspaces

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = W$$

by setting $W_r = \mathfrak{sp}(r) \cap W \, (1 \leq r \leq n)$. (Note that the numbering of the above chain is the reverse order of that in [4, p.79].) It is obvious that $W_r$ is a pseudo-abelian subspace of $\mathfrak{sp}(r)$ with respect to $\mathfrak{h}(r)^i$. Put

$$C_r = \big\{ x^r \in M(n,1;\mathbb{H}) \mid (x^1, \ldots, x^r, \overbrace{0, \ldots, 0}^{n-r}) \in W_r \big\} \quad (r = 1, \ldots, n).$$

Then it is clear that $C_r \cong W_r / W_{r-1} \, (1 \leq r \leq n)$ and $\dim W = c_1 + \cdots + c_n$, where we set $c_r = \dim C_r \, (1 \leq r \leq n)$. Moreover, by (1.2) and (1.3) we have

$$\big(C_p, C_q\big) = 0 \quad (1 \leq p < q \leq n), \tag{1.5}$$

$$\big(C_r, C_r\big) \subset \mathbb{C}^i \quad (1 \leq r \leq n). \tag{1.6}$$

The above equalities (1.5) and (1.6) will play decisive roles in the proof of Theorem 4.

By $C_r^{\mathbb{H}} \, (1 \leq r \leq n)$ we denote the right $\mathbb{H}$-subspace of $M(n,1;\mathbb{H})$ generated by $C_r$. Set $k_r = \dim_{\mathbb{H}} C_r^{\mathbb{H}} \, (1 \leq r \leq n)$. Then, in view of (1.5) and (1.4) we have

$$\big(C_p^{\mathbb{H}}, C_q^{\mathbb{H}}\big) = 0 \qquad (1 \leq p < q \leq n). \tag{1.7}$$

Utilizing (1.6) and (1.7), we have proved in [4] the following

**Lemma 5** (see [4]). *Under the setting stated above the following* (1) *and* (2) *hold*:

(1) $k_1 + \cdots + k_n \leq n$.

(2) $c_r \leq 2k_r \quad (1 \leq r \leq n)$.

*In particular, if* $\dim W = p_{Sp(n)} \, (= 2n)$, *then* $k_1 + \cdots + k_n = n$ *and* $c_r = 2k_r \, (1 \leq r \leq n)$.

In what follows we assume that $W$ is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying $\dim W = p_{Sp(n)}$. Let us define an $\mathbb{R}$-linear endomorphism $\xi \longmapsto \widetilde{\xi}$ of $M(n,1;\mathbb{H})$ by setting $\widetilde{\xi} = {}^t(\xi_1, \ldots, \xi_{n-1}, 0)$ for $\xi = {}^t(\xi_1, \ldots, \xi_n) \in M(n,1;\mathbb{H})$. Let $\widetilde{C_n}$ be the image of $C_n$ by this endomorphism. We first prove

**Lemma 6.** $k_n \geq 1$ *and* $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$.

**Proof.** Suppose that $k_n = 0$. Then we have $C_n = 0$ and hence $W = W_{n-1}$. Therefore, in a natural way $W$ may be regarded as a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. This implies $\dim W \leq p_{Sp(n-1)} = 2(n-1)$, contradicting the assumption $\dim W = 2n$. Consequently, we have $k_n \geq 1$. Let $\xi \in C_n$ and $\eta \in C_1 + \cdots + C_{n-1}$. Since $\eta$ is written as $\eta = {}^t(\eta_1, \ldots, \eta_{n-1}, 0)$, we have $\big(\widetilde{\xi}, \eta\big) = \big(\xi, \eta\big) = 0$ (see (1.5)). Hence we have $\big(\widetilde{C_n}, C_1 + \cdots + C_{n-1}\big) = 0$. Viewing (1.4), we have $\big(\widetilde{C_n}^{\mathbb{H}}, C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}\big) = 0$. Since both $\widetilde{C_n}^{\mathbb{H}}$ and $C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}$ may be regarded as subspaces of $M(n-1, 1; \mathbb{H})$, we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq n - 1 - (k_1 + \cdots + k_{n-1})$ (see (1.7)). Therefore by Lemma 5 we obtain $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$. □

Let $C_n'$ be the subset of $C_n$ consisting of all ${}^t(\xi_1, \ldots, \xi_n) \in C_n$ such that the $n$-th component $\xi_n \in \mathbb{D}^i$, i.e., $C_n' = \{{}^t(\xi_1, \ldots, \xi_n) \in C_n \mid \xi_n \in \mathbb{D}^i\}$. Clearly, $C_n'$ is a subspace of $C_n$. We denote by $\widetilde{C_n'}$ the image of $C_n'$ by the endomorphism $\xi \longmapsto \widetilde{\xi}$. Then we can show

**Lemma 7.** $\dim C_n' \geq 2k_n - 1$ *and* $\dim \widetilde{C_n'} \leq 2(k_n - 1)$.

**Proof.** First we note that $\xi_n \in \mathbb{R}e^i + \mathbb{D}^i$ holds for any $\xi = {}^t(\xi_1, \ldots, \xi_n) \in C_n$. Indeed, $\xi_n$ is the $(n,n)$-component of a certain matrix $X \in \mathfrak{sp}(n)$ (recall the definition of $C_n$). Consequently, we have $\dim C_n' \geq \dim C_n - 1 = c_n - 1 = 2k_n - 1$.

We next prove the second inequality. Let $\xi = {}^t(\xi_1, \ldots, \xi_n) \in C_n'$ and $\eta = {}^t(\eta_1, \ldots, \eta_n) \in C_n'$. Then we easily have $\big(\widetilde{\xi}, \widetilde{\eta}\big) = \big(\xi, \eta\big) - \overline{\xi_n}\eta_n$. Since $\big(\xi, \eta\big) \in \mathbb{C}^i$ (see (1.6)) and $\overline{\xi_n}\eta_n \in \mathbb{D}^i\mathbb{D}^i = \mathbb{C}^i$, it follows that $\big(\widetilde{\xi}, \widetilde{\eta}\big) \in \mathbb{C}^i$. This proves $\big(\widetilde{C_n'}, \widetilde{C_n'}\big) \subset \mathbb{C}^i$. By this fact we can deduce that $\widetilde{C_n'} \cap \widetilde{C_n'}e^j = 0$ for any $j\ (= 1, 2, 3)$ such that $j \neq i$. In fact, if there is an element $\widetilde{\xi} \in \widetilde{C_n'}$ such that $\widetilde{\xi}e^j \in \widetilde{C_n'}$, then we have $\mathbb{C}^i \ni \big(\widetilde{\xi}, \widetilde{\xi}e^j\big) = \big(\widetilde{\xi}, \widetilde{\xi}\big)e^j \in \mathbb{C}^i e^j = \mathbb{D}^i$. Since $\mathbb{C}^i \cap \mathbb{D}^i = 0$, it follows that $\big(\widetilde{\xi}, \widetilde{\xi}\big) = 0$, i.e., $\widetilde{\xi} = 0$. Thus, we know that $\widetilde{C_n'} + \widetilde{C_n'}e^j\ (\subset \widetilde{C_n}^{\mathbb{H}})$ is a direct sum if $j \neq i$. Consequently, we have $2\dim \widetilde{C_n'} \leq 4\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq 4(k_n - 1)$, i.e., $\dim \widetilde{C_n'} \leq 2(k_n - 1)$ (see Lemma 6). This completes the proof of the lemma. □

With the basis of Lemma 7 we can show

**Lemma 8.** *Let* $D_n$ *be the kernel of the linear mapping* $C_n \ni \xi \longmapsto \widetilde{\xi} \in \widetilde{C_n}$. *Then:*

(1) $D_n = \{{}^t(0, \ldots, 0, w) \in M(n, 1; \mathbb{H}) \mid w \in \mathbb{D}^i\}$.
(2) $\widetilde{C_n} \subset C_n$.
(3) $C_n = D_n + \widetilde{C_n}$ (*direct sum*); $\dim \widetilde{C_n} = c_n - 2$.

**Proof.** By Lemma 7 we have $\dim C_n' - \dim \widetilde{C_n'} \geq 2k_n - 1 - 2(k_n - 1) > 0$. This implies that $D_n \cap C_n' \neq 0$. Let $\xi$ be a non-trivial element of $D_n \cap C_n'$. Then, by the definitions of $D_n$ and $C_n'$, we know that $\xi$ may be written as $\xi = {}^t(0, \ldots, 0, w)$, where $w \in \mathbb{D}^i\ (w \neq 0)$. Let $\eta = {}^t(\eta_1, \ldots, \eta_n)$ be an arbitrary element of $C_n$. Then by (1.6) we have $\big(\xi, \eta\big) = \bar{w}\eta_n \in \mathbb{C}^i$.

Hence we can easily show that $\eta_n \in \mathbb{D}^i$ (see the proof for the case $n = 1$). Accordingly, $\eta \in C'_n$ and hence $C'_n = C_n$. Therefore, we have

$$\dim D_n = \dim C_n - \dim \widetilde{C_n} = \dim C_n - \dim \widetilde{C'_n} \geq c_n - 2(k_n - 1) = 2.$$

On the other hand, since $D_n \subset C_n = C'_n$, we have $D_n \subset \{{}^t(0, \ldots, 0, w) |\, w \in \mathbb{D}^i\}$ and hence $\dim D_n \leq \dim \mathbb{D}^i = 2$. This, together with the above inequality, proves $\dim D_n = 2$ and $D_n = \{{}^t(0, \ldots, 0, w) |\, w \in \mathbb{D}^i\}$. Thus we obtain (1).

Let $\zeta = {}^t(\zeta_1, \ldots, \zeta_n) \in M(n, 1; \mathbb{H})$ be an arbitrary element of $C_n$. Since $C_n = C'_n$, we have $\zeta_n \in \mathbb{D}^i$ and hence $\zeta' = {}^t(0, \ldots, 0, \zeta_n) \in D_n \subset C_n$. Consequently, $\widetilde{\zeta} = {}^t(\zeta_1, \ldots, \zeta_{n-1}, 0) = \zeta - \zeta' \in C_n$, showing (2). The assertion (3) immediately follows from (1) and (2). □

With these preparations we can show

**Lemma 9.** $\widetilde{C_n} = 0$. *Accordingly,* $C_n = D_n$.

**Proof.** We first prove

$$\widetilde{C_n} \cap \widetilde{C_n} e^i = 0. \tag{1.8}$$

Suppose that there is an element $\widetilde{\xi} = {}^t(\xi_1, \ldots, \xi_{n-1}, 0) \in \widetilde{C_n}$ such that $\widetilde{\xi} e^i \in \widetilde{C_n}$. Note that $\widetilde{C_n} \subset C_n$ (see Lemma 8 (2)). By the definition of $C_n$ we know that there are matrices $X$ and $Y \in W$ written in the form

$$X = \begin{pmatrix} X' & \xi' \\ -{}^t\bar{\xi}' & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y' & \xi' e^i \\ e^i\, {}^t\bar{\xi}' & 0 \end{pmatrix},$$

where $X'$, $Y' \in \mathfrak{sp}(n-1)$ and $\xi' = {}^t(\xi_1, \ldots, \xi_{n-1}) \in M(n-1, 1; \mathbb{H})$. Take an integer $j\,(= 1, 2, 3)$ such that $j \neq i$. Since ${}^t(0, \ldots, 0, e^j) \in D_n \subset C_n$, we know that there is an element $Z \in W$ of the form

$$Z = \begin{pmatrix} Z' & 0 \\ 0 & e^j \end{pmatrix},$$

where $Z' \in \mathfrak{sp}(n-1)$. Since $W$ is a pseudo-abelian with respect to $\mathfrak{h}(n)^i$, we have $[X, Z] \in \mathfrak{h}(n)^i$ and $[Y, Z] \in \mathfrak{h}(n)^i$. Hence by a direct calculation we can show

$$Z'\xi' = \xi' e^j; \qquad Z'(\xi' e^i) = (\xi' e^i) e^j. \tag{1.9}$$

By the second equality of (1.9) we have $(Z'\xi')e^i = \xi'(e^i e^j) = -\xi'(e^j e^i) = -(\xi' e^j)e^i$ and hence $Z'\xi' = -\xi' e^j$. This, together with the first equality of (1.9), proves $Z'\xi' = \xi' e^j = 0$. Hence we have $\xi' = 0$, i.e., $\widetilde{\xi} = 0$. This implies (1.8). As a result of (1.8), the subspace $\widetilde{C_n} + \widetilde{C_n} e^i$ $(\subset \widetilde{C_n}^{\mathbb{H}})$ is a direct sum. Since $\dim \widetilde{C_n} = c_n - 2 = 2(k_n - 1)$ (see Lemma 8 (3) and Lemma 5), it follows that $\dim_{\mathbb{R}} \widetilde{C_n}^{\mathbb{H}} \geq 2 \dim \widetilde{C_n} = 4(k_n - 1)$. Hence we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} = (1/4) \dim_{\mathbb{R}} \widetilde{C_n}^{\mathbb{H}} \geq k_n - 1$. On the other hand, we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$ (see Lemma 6).

Therefore, we obtain $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} = k_n - 1$ and $\widetilde{C_n}^{\mathbb{H}} = \widetilde{C_n} + \widetilde{C_n} e^i$. More strongly, we can prove $\widetilde{C_n} = 0$. In fact, since $\widetilde{C_n}^{\mathbb{H}} = \widetilde{C_n} + \widetilde{C_n} e^i$, it follows that

$$\big(\widetilde{C_n}^{\mathbb{H}}, \widetilde{C_n}^{\mathbb{H}}\big) \subset \big(\widetilde{C_n}, \widetilde{C_n}\big) + \big(\widetilde{C_n} e^i, \widetilde{C_n}\big) + \big(\widetilde{C_n}, \widetilde{C_n} e^i\big) + \big(\widetilde{C_n} e^i, \widetilde{C_n} e^i\big).$$

If $\widetilde{C_n} \neq 0$, then it is easy to see that $\big(\widetilde{C_n}^{\mathbb{H}}, \widetilde{C_n}^{\mathbb{H}}\big) = \mathbb{H}$. However, the right side of the above inclusion is contained in $\mathbb{C}^i$, because $\big(\widetilde{C_n}, \widetilde{C_n}\big) \subset \big(C_n, C_n\big) \subset \mathbb{C}^i$ (see Lemma 8 (2) and (1.6)), $\big(\widetilde{C_n} e^i, \widetilde{C_n}\big) \subset e^i \mathbb{C}^i = \mathbb{C}^i$, $\big(\widetilde{C_n}, \widetilde{C_n} e^i\big) \subset \mathbb{C}^i e^i = \mathbb{C}^i$ and $\big(\widetilde{C_n} e^i, \widetilde{C_n} e^i\big) \subset e^i \mathbb{C}^i e^i = \mathbb{C}^i$ (see (1.4)). This is a contradiction. Hence we have $\widetilde{C_n} = 0$. The equality $C_n = D_n$ now follows immediately. □

**Proof of Theorem** 4**.** By Lemma 9 and Lemma 8 (3) we have $c_n = 2k_n = 2$. Hence, $W_{n-1}$, which is a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$, satisfies $\dim W_{n-1} = c_1 + \cdots + c_{n-1} = 2(n-1) = p_{Sp(n-1)}$. Therefore, by the hypothesis of our induction we know that $W_{n-1} = \mathfrak{p}(n-1)^i$. From this fact we can deduce $W = \mathfrak{p}(n)^i$. In fact, let $X$ be an arbitrary element of $W$. Then $X$ may be written as $X = \left(\begin{smallmatrix} X' & 0 \\ 0 & w \end{smallmatrix}\right)$, where $X' \in \mathfrak{sp}(n-1)$, $w \in \mathbb{D}^i$ (see Lemma 9 and Lemma 8 (1)). Since $[X, W_{n-1}] \subset \mathfrak{h}(n)^i$, it follows that $[X', \mathfrak{p}(n-1)^i] \subset \mathfrak{h}(n-1)^i$. Hence we have $X' \in \mathfrak{p}(n-1)^i$, because $\mathfrak{p}(n-1)^i$ is a maximal pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. Consequently, we have $X \in \mathfrak{p}(n)^i$ and $W = \mathfrak{p}(n)^i$, which completes the proof of Theorem 4. □

## 2. The Gauss equation of $Sp(n)$

Let $M$ be a Riemannian manifold. We denote by $g$ the Riemannian metric of $M$ and by $R$ the Riemannian curvature tensor of type $(1,3)$ with respect to $g$. Let $x \in M$ and let $T_x(M)$ (resp. $T_x^*(M)$) be the tangent (resp. cotangent) vector space of $M$ at $x$. Let $r$ be a non-negative integer. We define a quadratic equation with respect to an unknown $\mathbf{\Psi} \in S^2 T_x^*(M) \otimes \mathbb{R}^r$ by

$$-g(R(X,Y)Z,W) = \big\langle \mathbf{\Psi}(X,Z), \mathbf{\Psi}(Y,W)\big\rangle - \big\langle \mathbf{\Psi}(X,W), \mathbf{\Psi}(Y,Z)\big\rangle, \tag{2.1}$$

where $X$, $Y$, $Z$, $W \in T_x(M)$ and $\langle\, , \rangle$ is the standard inner product of $\mathbb{R}^r$. We call (2.1) the *Gauss equation* in codimension $r$ at $x$. The set of solutions of (2.1) is called the *Gaussian variety* in codimension $r$ at $x$ and is denoted by $\mathcal{G}_x(M, \mathbb{R}^r)$.

Let $O(r)$ be the orthogonal group of $\mathbb{R}^r$. We define an action of $O(r)$ on $S^2 T_x^*(M) \otimes \mathbb{R}^r$ by

$$(\rho \mathbf{\Psi})(X,Y) = \rho(\mathbf{\Psi}(X,Y)), \quad X, Y \in T_x(M),\ \rho \in O(r). \tag{2.2}$$

As is easily seen, if $\mathbf{\Psi}$ is a solution of (2.1), then $\rho\mathbf{\Psi}$ is also a solution of (2.1) for any $\rho \in O(r)$. We say that $\mathcal{G}_x(M, \mathbb{R}^r)$ is *EOS* if $\mathcal{G}_x(M, \mathbb{R}^r) \neq \emptyset$ and if $\mathcal{G}_x(M, \mathbb{R}^r)$ is composed of essentially one solution, i.e., for any solutions $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2 \in \mathcal{G}_x(M, \mathbb{R}^r)$ there is an element $\rho \in O(r)$ such that $\mathbf{\Psi}_2 = \rho \mathbf{\Psi}_1$.

In the following we consider the case where $M$ is the the symplectic group $Sp(n)$ endowed with the bi-invariant metric $\nu$, which is induced from the inclusion $Sp(n) \subset M(n,n;\mathbb{H})$. As usual we identify the tangent space of $Sp(n)$ at the identity $I_n$ with the Lie algebra $\mathfrak{sp}(n)$. We denote by $(\ ,\ )$ the inner product of $\mathfrak{sp}(n)$ induced from $\nu$ at $I_n$. The curvature transformation $R_0(X,Y)\,(X,Y \in \mathfrak{sp}(n))$ of $Sp(n)$ at $I_n$ is given by $R_0(X,Y) = -\frac{1}{4}\mathrm{ad}\,([X,Y])$ (see [14]). Hence at $I_n$ the Gauss equation (2.1) is written as

$$\frac{1}{4}\big([[X,Y],Z],W\big) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle - \big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle, \tag{2.3}$$

where $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathbb{R}^r$ and $X,Y,Z,W \in \mathfrak{sp}(n)$. We simply denote by $\mathcal{G}(Sp(n),\mathbb{R}^r)$ the Gaussian variety in codimension $r$ at $I_n$. The main aim of this and the subsequent sections is to prove

**Theorem 10.** *For any positive integer $n$ the Gaussian variety $\mathcal{G}(Sp(n),\mathbb{R}^{2n^2-n})$ in codimension $2n^2-n$ is EOS.*

By homogeneity, we know that the Gaussian variety $\mathcal{G}_x(Sp(n),\mathbb{R}^{2n^2-n})$ in codimension $2n^2-n$ is $EOS$ at each $x \in Sp(n)$. By this result we conclude that $Sp(n)$ is formally rigid in codimension $2n^2-n$. (For the definition of formal rigidness, see [8].) Accordingly, by Theorem 5 of [8] we can establish the rigidity theorem of $Sp(n)$ (Theorem 1).

In the following we will prove Theorem 10 by induction on $n$. As we have stated in the introduction, if $n=1$, then $Sp(1) \cong S^3$ and the canonical isometric imbedding $\boldsymbol{f}_0$ is the inclusion map of the standard sphere $S^3$ with radius 1 into $\mathbb{R}^4$. The second fundamental form $\boldsymbol{\Psi}_0$ of $\boldsymbol{f}_0$ at $\boldsymbol{x} \in S^3$ is given by $\boldsymbol{\Psi}_0 = -\nu\boldsymbol{x}$. Hence $\boldsymbol{f}_0$ is a typical example of an isometric imbedding with type number 3. By applying the theory of type number in [12] or by a direct calculation we know that any solution $\boldsymbol{\Psi}$ of the Gauss equation of $S^3$ in codimension 1 can be represented by $\boldsymbol{\Psi} = \pm\boldsymbol{\Psi}_0$. Therefore we get Theorem 10 for the case $n=1$. For this reason we may assume $n \geq 2$ in the following discussion.

**Remark 11.** It should be noted that in case $n \geq 2$ the theory of type number in [12] is not applicable to the canonical isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)$. In fact, for an isometric imbedding $\boldsymbol{f}$ of a Riemannian manifold $M$ into the euclidean space $\mathbb{R}^m$, the type number $k$ of $\boldsymbol{f}$ must satisfy the inequality $k \leq \dim M/(m - \dim M)$ (see [18] or [16]). Consequently, in the case of $\boldsymbol{f}_0$ we can easily show that $k < 2$ when $n \geq 2$.

Now let $\mathfrak{N}(n)$ be the subspace of $M(n,n;\mathbb{H})$ composed of all $X \in M(n,n;\mathbb{H})$ satisfying ${}^t\bar{X} = X$. Clearly, we have $\dim \mathfrak{N}(n) = 2n^2-n$ and

$$M(n,n;\mathbb{H}) = \mathfrak{sp}(n) + \mathfrak{N}(n) \qquad \text{(orthogonal direct sum)}.$$

As is easily seen, $\mathfrak{N}(n)$ is the normal vector space of the canonical isometric imbedding $\boldsymbol{f}_0$ at $I_n$. The second fundamental form $\boldsymbol{\Psi}_0$ of $\boldsymbol{f}_0$ at $I_n$ is an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$

given by

$$\boldsymbol{\Psi}_0(X,Y) = \frac{1}{2}\big(XY + YX\big), \quad X, Y \in \mathfrak{sp}(n) \tag{2.4}$$

(see [15, p.370]). Under a natural identification $(\mathfrak{N}(n), \nu) \cong (\mathbb{R}^{2n^2-n}, \langle\ ,\ \rangle)$ as euclidean vector spaces we can regard the unknown $\boldsymbol{\Psi}$ in the Gauss equation (2.3) in codimension $2n^2 - n$ as an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. (In what follows, the inner product $\nu$ of $\mathfrak{N}(n)$ will be denoted by $\langle\ ,\ \rangle$.) Therefore the Gaussian variety $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ may be considered as a subset of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. In this meaning we write $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ as $\mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then $\boldsymbol{\Psi}_0$ may be considered as an element of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$, which is called the *canonical solution* of the Gauss equation (2.3) in codimension $2n^2 - n$. Now Theorem 10 may be stated in the following way: Any solution $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ of the Gauss equation (2.3) is equivalent to $\boldsymbol{\Psi}_0$, i.e., there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\boldsymbol{\Psi} = \rho\boldsymbol{\Psi}_0$, where $O(\mathfrak{N}(n))$ stands for the orthogonal group of $\mathfrak{N}(n)$.

## 3. The space $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$

In this section we assume that $n \geq 2$. Let $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $X \in \mathfrak{sp}(n)$. We define a linear mapping $\boldsymbol{\Psi}_X : \mathfrak{sp}(n) \longrightarrow \mathfrak{N}(n)$ by setting $\boldsymbol{\Psi}_X(Y) = \boldsymbol{\Psi}(X,Y)$ $(Y \in \mathfrak{sp}(n))$. By $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ $(\subset \mathfrak{sp}(n))$ we denote the kernel of $\boldsymbol{\Psi}_X$. In this section we investigate the kernel $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ for a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.3), i.e., $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. As in the case of $P^2(\mathbb{H})$ or $P^2(\mathbb{CAY})$, the knowledge about $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ will play an important role to determine the solutions of the Gauss equation (2.3) (cf. [8] and [9]).

Let $X \in \mathfrak{sp}(n)$. By $C(X)$ we denote the centralizer of $X$ in $\mathfrak{sp}(n)$. Then we have

**Lemma 12.** *Let $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and $X \in \mathfrak{sp}(n)$. Then:*

(1) $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(X) \geq 2n$.

(2) *If $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, then $[\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)] \subset C(X)$.*

**Proof.** Since

$$\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(X) \geq \dim Sp(n) - \dim \mathfrak{N}(n) = (2n^2 + n) - (2n^2 - n) = 2n,$$

we get (1). Assume that $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then by (2.3) for each $Y \in \mathfrak{sp}(n)$ we have

$$\big([[\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)], X], Y]\big) \subset \big\langle \boldsymbol{\Psi}(\boldsymbol{K}_{\boldsymbol{\Psi}}(X), X), \boldsymbol{\Psi}(\boldsymbol{K}_{\boldsymbol{\Psi}}(X), Y)\big\rangle = 0.$$

Consequently, we have $[[\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)], X] = 0$. The assertion (2) immediately follows from this equality (cf. [10, Lemma 3]). □

Let $X \in \mathfrak{sp}(n)$. Since $\mathfrak{sp}(n)$ is a compact simple Lie algebra, we know that $\dim C(X) \geq \mathrm{rank}(\mathfrak{sp}(n)) = n$. We recall that an element $X \in \mathfrak{sp}(n)$ is called *regular* (resp. *singular*) if $\dim C(X) = n$ (resp. $\dim C(X) > n$).

**Lemma 13.** *Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ and $H \in \mathfrak{h}(n)^i$ $(i = 1, 2, 3)$. Then $\boldsymbol{K}_{\boldsymbol{\Psi}}(H) \supset \mathfrak{p}(n)^i$. If $H$ is regular, then the equality $\boldsymbol{K}_{\boldsymbol{\Psi}}(H) = \mathfrak{p}(n)^i$ holds.*

**Proof.** Let $H \in \mathfrak{h}(n)^i$. Then by Lemma 12 (2) we have $[\boldsymbol{K}_{\boldsymbol{\Psi}}(H), \boldsymbol{K}_{\boldsymbol{\Psi}}(H)] \subset C(H)$. Assume that $H$ is regular. Then, since $C(H) = \mathfrak{h}(n)^i$, we have $[\boldsymbol{K}_{\boldsymbol{\Psi}}(H), \boldsymbol{K}_{\boldsymbol{\Psi}}(H)] \subset \mathfrak{h}(n)^i$. This implies that $\boldsymbol{K}_{\boldsymbol{\Psi}}(H)$ is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$. Therefore we have $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H) \leq p_{Sp(n)} = 2n$ (see Theorem 2). On the other hand, since $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H) \geq 2n$ (see Lemma 12 (1)), it follows that $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H) = 2n$. Hence $\boldsymbol{K}_{\boldsymbol{\Psi}}(H) = \mathfrak{p}(n)^i$ (see Theorem 4). Let $H' \in \mathfrak{h}(n)^i$ be an arbitrary element. Note that regular elements are dense in $\mathfrak{h}(n)^i$ and, as we have shown, $\boldsymbol{\Psi}(H, \mathfrak{p}(n)^i) = 0$ holds for any regular element $H \in \mathfrak{h}(n)^i$. Because of the continuity of $\boldsymbol{\Psi}$ we have $\boldsymbol{\Psi}(H', \mathfrak{p}(n)^i) = 0$. This shows that $\boldsymbol{K}_{\boldsymbol{\Psi}}(H') \supset \mathfrak{p}(n)^i$. □

Let $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. We define an element $\boldsymbol{\Psi}^g \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ by

$$(\boldsymbol{\Psi}^g)(X, Y) = \boldsymbol{\Psi}(\mathrm{Ad}(g^{-1})X, \mathrm{Ad}(g^{-1})Y), \quad X, Y \in \mathfrak{sp}(n). \tag{3.1}$$

Then we can easily see the following

**Lemma 14.** *Let $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. Then:*

(1) $\boldsymbol{K}_{\boldsymbol{\Psi}^g}(X) = \mathrm{Ad}(g)\boldsymbol{K}_{\boldsymbol{\Psi}}(\mathrm{Ad}(g^{-1})X), \quad X \in \mathfrak{sp}(n)$.

(2) $\boldsymbol{\Psi}^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ *if and only if* $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

Combining Lemma 13 with Lemma 14, we have

**Proposition 15.** *Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, $X \in \mathfrak{sp}(n)$ and $g \in Sp(n)$. Assume that $\mathrm{Ad}(g)X \in \mathfrak{h}(n)^i$ for some $i\,(= 1, 2, 3)$. Then $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) \supset \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. Further, if $X$ is regular, then $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$.*

**Proof.** Note that $\boldsymbol{\Psi}^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ (see Lemma 14 (2)). Applying Lemma 13 to $\boldsymbol{\Psi}^g$, we have $\boldsymbol{K}_{\boldsymbol{\Psi}^g}(\mathrm{Ad}(g)X) \supset \mathfrak{p}(n)^i$. Therefore by Lemma 14 (1) we have $\mathfrak{p}(n)^i \subset \boldsymbol{K}_{\boldsymbol{\Psi}^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$. Consequently, $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) \supset \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. If $X$ is regular, then $\mathrm{Ad}(g)X$ is also regular. Accordingly, we have $\boldsymbol{K}_{\boldsymbol{\Psi}^g}(\mathrm{Ad}(g)X) = \mathfrak{p}(n)^i$ and hence $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. □

**Remark 16.** Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. It is well-known that any element of $\mathfrak{sp}(n)$ is conjugate to an element of a Cartan subalgebra $\mathfrak{h}(n)^i$. Therefore, for a regular element $X \in \mathfrak{sp}(n)$ the space $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ is determined by Proposition 15. Here we note that if $X$ is regular, then $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ does not depend on the choice of the solution $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, i.e., $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \boldsymbol{K}_{\boldsymbol{\Psi}'}(X)$ holds for any $\boldsymbol{\Psi}$, $\boldsymbol{\Psi}' \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

In the following discussion, we will determine $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ for singular elements $X \in \mathfrak{sp}(n)$ of special type. By Proposition 15 we immediately obtain

**Proposition 17.** *Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Let $i = 1$, $2$ or $3$ and $X \in \mathfrak{sp}(n)$. Denote by $G_X^i$ the subset of $Sp(n)$ consisting of all $g \in Sp(n)$ such that $\mathrm{Ad}(g)X \in \mathfrak{h}(n)^i$. Then:*

$$\boldsymbol{K}_{\boldsymbol{\Psi}}(X) \supset \sum_{g \in G_X^i} \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i. \tag{3.2}$$

Let $a$, $b$ and $i$ are integers satisfying $1 \leq a \neq b \leq n$, $1 \leq i \leq 3$. Define elements $H_a^i$, $P_{ab}$ and $Q_{ab}^i \in M(n,n;\mathbb{H})$ by

$$H_a^i = E_{aa}e^i; \quad P_{ab} = -P_{ba} = E_{ab} - E_{ba}; \quad Q_{ab}^i = Q_{ba}^i = (E_{ab} + E_{ba})e^i.$$

Then it is easily seen that $H_a^i$, $P_{ab}$, $Q_{ab}^i \in \mathfrak{sp}(n)$ and

$$\begin{aligned} \big(H_a^i, H_b^j\big) &= \delta_{ab}\delta_{ij}; \quad \big(H_a^i, P_{cd}\big) = \big(H_a^i, Q_{cd}^j\big) = 0; \\ \big(P_{ab}, P_{cd}\big) &= 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}); \quad \big(P_{ab}, Q_{cd}^i\big) = 0; \\ \big(Q_{ab}^i, Q_{cd}^j\big) &= 2(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\delta_{ij}. \end{aligned} \tag{3.3}$$

Therefore the set $\{H_a^i\,(1 \leq a \leq n)\}$ forms an orthonormal basis of $\mathfrak{h}(n)^i\,(1 \leq i \leq 3)$ and the set $\{H_a^i\,(1 \leq a \leq n, 1 \leq i \leq 3),\ (1/\sqrt{2})P_{ab}\,(1 \leq a < b \leq n),\ (1/\sqrt{2})Q_{ab}^i\,(1 \leq a < b \leq n, 1 \leq i \leq 3)\}$ forms an orthonormal basis of $\mathfrak{sp}(n)$.

Let $a$, $b$ and $i$ are integers satisfying $1 \leq a \neq b \leq n$, $1 \leq i \leq 3$. Define a subspace $\mathfrak{s}_{ab}^i$ by $\mathfrak{s}_{ab}^i = \mathbb{R}(H_a^i - H_b^i) + \mathbb{R}P_{ab} + \mathbb{R}Q_{ab}^i$. By an easy calculation we have

$$[H_a^i - H_b^i, P_{ab}] = 2Q_{ab}^i; \qquad [H_a^i - H_b^i, Q_{ab}^i] = -2P_{ab};$$
$$[P_{ab}, Q_{ab}^i] = 2(H_a^i - H_b^i).$$

This indicates that $\mathfrak{s}_{ab}^i$ forms a three-dimensional subalgebra of $\mathfrak{sp}(n)$ and is not abelian. Now we note the following lemma, which holds for any compact Lie algebra:

**Lemma 18.** *Let $\mathfrak{s}$ be a three-dimensional subalgebra of a compact Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{s}$ is not abelian. Then, for any linearly independent elements $Z$, $Z' \in \mathfrak{s}$, there is an element $g \in \exp(\mathbb{R}[Z,Z'])$ $(\subset \exp(\mathfrak{g}))$ such that $\mathrm{Ad}(g)Z = \mathbb{R}Z'$.*

**Proof.** Since $\mathfrak{g}$ is compact, $\mathfrak{s}$ is also a compact Lie algebra. Hence $\mathfrak{s}$ may be represented by a direct sum of its center and its semi-simple part. Note that any simple Lie algebra is of dimension $\geq 3$. Under the assumption that $\mathfrak{s}$ is not abelian and $\dim \mathfrak{s} = 3$, we know that the center of $\mathfrak{s}$ is trivial and that $\mathfrak{s}$ is simple. Hence, $\mathfrak{s}$ is isomorphic to the simple Lie algebra $\mathfrak{su}(2)$.

Let $B$ be an $\mathrm{ad}\,(\mathfrak{g})$-invarinat inner product of $\mathfrak{g}$. Let $Z$, $Z' \in \mathfrak{s}$. If $Z$ and $Z'$ are linearly independent, then it follows that $[Z,Z'] \neq 0$, because $\mathrm{rank}(\mathfrak{s}) = 1$. Set $\mathfrak{s}' = \mathbb{R}Z + \mathbb{R}Z'$. Then we have $B(\mathfrak{s}', \mathbb{R}[Z,Z']) = 0$, i.e., $\mathbb{R}[Z,Z']$ is the orthogonal complement of $\mathfrak{s}'$ in $\mathfrak{s}$ with respect to $B$. Indeed, we have

$$B(Z,[Z,Z']) = B([Z,Z],Z') = 0; \quad B(Z',[Z,Z']) = -B([Z',Z'],Z) = 0.$$

Similarly, we can prove $B(\mathrm{ad}\,[Z,Z'](Z),[Z,Z']) = B(\mathrm{ad}\,[Z,Z'](Z'),[Z,Z']) = 0$. This means that $\mathfrak{s}'$ is invariant by $\mathrm{ad}\,[Z,Z']$. Moreover, we have $\mathrm{ad}\,([Z,Z'])Z'' \neq 0$ for any $Z'' \in \mathfrak{s}'$ with $Z'' \neq 0$. Therefore, $\mathrm{Ad}(\exp(\mathbb{R}[Z,Z']))$ forms a non-trivial subgroup of rotations of $\mathfrak{s}'$ with respect to $B$. From this fact the lemma follows immediately. $\square$

In the following, we say a subalgebra $\mathfrak{s}$ of $\mathfrak{sp}(n)$ is $NAT$ if $\mathfrak{s}$ is non-abelian and $\dim \mathfrak{s} = 3$. As we have seen, $\mathfrak{s}^i_{ab} = \mathbb{R}(H^i_a - H^i_b) + \mathbb{R}P_{ab} + \mathbb{R}Q^i_{ab}$ is $NAT$. For non-zero elements $X$ and $Y \in \mathfrak{sp}(n)$ we write $X \sim Y$ if there is an element $g \in Sp(n)$ such that $\mathrm{Ad}(g)X \in \mathbb{R}Y$. Apparently, $\sim$ defines an equivalence relation in $\mathfrak{sp}(n)\setminus\{0\}$. According to Lemma 18 if $\mathfrak{s}$ is $NAT$, then $Z \sim Z'$ for any $Z, Z' \in \mathfrak{s}\setminus\{0\}$. For example, we have $(H^i_a - H^i_b) \sim P_{ab} \sim Q^i_{ab}$.

For simplicity in the following discussion we set $\boldsymbol{K}_0(X) = \boldsymbol{K}_{\Psi_0}(X)$. As in the previous section we regard $\mathfrak{sp}(s)$ $(0 \leq s < n)$ as a subalgebra of $\mathfrak{sp}(n)$. Then by easy calculations we have

$$\begin{aligned} \boldsymbol{K}_0(H^i_n) &= \mathfrak{sp}(n-1) + \sum_{j\neq i} \mathbb{R}H^j_n; \\ \boldsymbol{K}_0(H^i_{n-1} + H^i_n) &= \mathfrak{sp}(n-2) + \sum_{j\neq i} \mathbb{R}H^j_{n-1} + \sum_{j\neq i} \mathbb{R}H^j_n + \sum_{j\neq i} \mathbb{R}Q^j_{n-1,n}. \end{aligned} \tag{3.4}$$

Let $\Psi$ be an arbitrary solution of the Gauss equation (2.3). By Remark 16 we know that $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_0(X)$ holds for a regular element $X \in \mathfrak{sp}(n)$. We now extend this relation to singular elements:

**Proposition 19.** *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then for each $i\,(= 1, 2, 3)$ it holds:*

(1) $\boldsymbol{K}_\Psi(H^i_n) = \boldsymbol{K}_0(H^i_n)$.
(2) $\boldsymbol{K}_\Psi(H^i_{n-1} + H^i_n) = \boldsymbol{K}_0(H^i_{n-1} + H^i_n)$.

**Proof.** Let $Sp(n-1)$ be the analytic subgroup of $Sp(n)$ corresponding to the subalgebra $\mathfrak{sp}(n-1)$. Let $g \in Sp(n-1)$. Then it is easy to see that $\mathrm{Ad}(g)H^i_n = H^i_n$. Hence by Proposition 17 we have $\boldsymbol{K}_\Psi(H^i_n) \supset \sum_{g\in Sp(n-1)} \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. Since $\mathfrak{h}(n-1)^j$ $(j \neq i)$ is a Cartan subalgebra of $\mathfrak{sp}(n-1)$, any element of $\mathfrak{sp}(n-1)$ is conjugate to an element of $\mathfrak{h}(n-1)^j$ under the action of $Sp(n-1)$. Hence we have $\bigcup_{g\in Sp(n-1)} \mathrm{Ad}(g^{-1})\mathfrak{h}(n-1)^j = \mathfrak{sp}(n-1)$. Since $\mathfrak{p}(n)^i \supset \mathfrak{h}(n-1)^j$, we have $\boldsymbol{K}_\Psi(H^i_n) \supset \mathfrak{sp}(n-1)$. This, together with $\boldsymbol{K}_\Psi(H^i_n) \supset \mathfrak{p}(n)^i$, shows $\boldsymbol{K}_\Psi(H^i_n) \supset \mathfrak{sp}(n-1) + \mathfrak{p}(n)^i = \boldsymbol{K}_0(H^i_n)$. We now show the equality $\boldsymbol{K}_\Psi(H^i_n) = \boldsymbol{K}_0(H^i_n)$. Take an element $X \in \boldsymbol{K}_\Psi(H^i_n) \cap \boldsymbol{K}_0(H^i_n)^\perp$, where $\boldsymbol{K}_0(H^i_n)^\perp$ is the orthogonal complement of $\boldsymbol{K}_0(H^i_n)$ in $\mathfrak{sp}(n)$. Then $X$ can be expressed as

$$X = \begin{pmatrix} 0 & \xi \\ -{}^t\bar{\xi} & c\,e^i \end{pmatrix}, \qquad \xi \in M(n-1, 1; \mathbb{H}),\ c \in \mathbb{R}.$$

Take $j,\ k\,(= 1, 2, 3)$ so that $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$. Then since $X \in \boldsymbol{K}_\Psi(H^i_n)$ and $H^j_n \in \boldsymbol{K}_\Psi(H^i_n)$, we obtain by Lemma 12 the following

$$0 = [[X, H^j_n], H^i_n] = \begin{pmatrix} 0 & -\xi\,e^k \\ -e^k\,{}^t\bar{\xi} & 4c\,e^j \end{pmatrix}.$$

Hence we have $\xi = 0$ and $c = 0$, i.e., $X = 0$. This proves $\boldsymbol{K}_\Psi(H^i_n) \cap \boldsymbol{K}_0(H^i_n)^\perp = 0$, i.e., $\boldsymbol{K}_\Psi(H^i_n) = \boldsymbol{K}_0(H^i_n)$.

Next we prove $\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)=\boldsymbol{K}_0(H^i_{n-1}+H^i_n)$. As in the case of $\boldsymbol{K}_{\Psi}(H^i_n)$, we can easily show that $\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)\supset\mathfrak{sp}(n-2)+\sum_{j\neq i}\mathbb{R}H^j_{n-1}+\sum_{j\neq i}\mathbb{R}H^j_n$. Take an element $Y\in\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)$ such that $\big(Y,\mathfrak{sp}(n-2)+\sum_{j\neq i}\mathbb{R}H^j_{n-1}+\sum_{j\neq i}\mathbb{R}H^j_n\big)=0$. Then $Y$ can be expressed as

$$Y=\begin{pmatrix}0&\xi&\eta\\-{}^t\bar{\xi}&\alpha&\beta\\-{}^t\bar{\eta}&-\bar{\beta}&\gamma\end{pmatrix},\qquad \xi,\eta\in M(n-2,1;\mathbb{H}),\ \alpha,\gamma\in\mathbb{R}e^i,\ \beta\in\mathbb{H}.$$

Take $j,\ k\,(=1,2,3)$ so that $\{i,j,k\}$ is an even permutation of $\{1,2,3\}$. Then by a direct calculation have

$$[[Y,H^j_{n-1}\pm H^j_n],H^i_{n-1}+H^i_n]=\begin{pmatrix}0&-\xi e^k&\mp\eta e^k\\-e^k\,{}^t\overline{\xi}&-4\alpha e^k&\beta''\\\mp e^k\,{}^t\overline{\eta}&-\overline{\beta''}&\mp4\gamma e^k\end{pmatrix},$$

where $\beta'=\pm\beta e^j-e^j\beta$, $\beta''=\beta'e^i-e^i\beta'$. (Note that $e^j\alpha=-\alpha e^j$, $e^j\gamma=-\gamma e^j$, $e^i\alpha=\alpha e^i$, $e^i\gamma=\gamma e^i$, because $\alpha,\ \gamma\in\mathbb{R}e^i$.) Since $Y\in\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)$ and $H^j_{n-1}\pm H^j_n\in\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)$, we have $[[Y,H^j_{n-1}\pm H^j_n],H^i_{n-1}+H^i_n]=0$ (see Lemma 12). Hence we conclude that $\xi=\eta=0$ and $\alpha=\gamma=0$ and $\beta''=0$. From the equality $\beta''=0$, we immediately have $\beta'\in\mathbb{C}^i$. Further, from $\beta'\in\mathbb{C}^i$ we can easily conclude that $\beta\in\mathbb{D}^i$. Thus we have $Y\in\sum_{j\neq i}\mathbb{R}Q^j_{n-1,n}$ and hence $\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)\subset\boldsymbol{K}_0(H^i_{n-1}+H^i_n)$.

To complete the proof of (2) we have to show $\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)\supset\sum_{j\neq i}\mathbb{R}Q^j_{n-1,n}$. Take $j\,(1\leq j\leq 3)$ such that $j\neq i$. Since $\mathfrak{s}^j_{n-1,n}=\mathbb{R}(H^j_{n-1}-H^j_n)+\mathbb{R}P_{n-1,n}+\mathbb{R}Q^j_{n-1,n}$ is $NAT$, there is an element $g\in\exp(\mathbb{R}P_{n-1,n})$ such that $\mathrm{Ad}(g)Q^j_{n-1,n}\in\mathbb{R}(H^j_{n-1}-H^j_n)\ (\subset\mathfrak{p}(n)^i)$ (see Lemma 18). Moreover, since $[P_{n-1,n},H^i_{n-1}+H^i_n]=0$, we have $\mathrm{Ad}(g)(H^i_{n-1}+H^i_n)=H^i_{n-1}+H^i_n\in\mathfrak{h}(n)^i$, i.e., $g\in G^i_{(H^i_{n-1}+H^i_n)}$. Therefore, by Proposition 17 we have $Q^j_{n-1,n}\in\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)$. Accordingly, it follows that $\boldsymbol{K}_{\Psi}(H^i_{n-1}+H^i_n)\supset\sum_{j\neq i}\mathbb{R}Q^j_{n-1,n}$, completing the proof of (2). $\square$

By $\mathcal{S}$ we denote the subset of $\mathfrak{sp}(n)$ consisting of all non-zero elements $X\in\mathfrak{sp}(n)$ such that $X\sim H^i_n$ or $X\sim H^i_{n-1}+H^i_n$ for some $i\,(=1,2,3)$. We note that each element $X\in\mathcal{S}$ is a singular element of $\mathfrak{sp}(n)$, because $H^i_n$ and $H^i_{n-1}+H^i_n$ are singular elements of $\mathfrak{sp}(n)$.

By use of Proposition 19 we can prove

**Proposition 20.** *Let $\Psi\in\mathcal{G}(Sp(n),\mathfrak{N}(n))$. Assume $X\in\mathcal{S}$. Then $\boldsymbol{K}_{\Psi}(X)=\boldsymbol{K}_0(X)$.*

**Proof.** Let $g\in Sp(n)$. Then we have $\Psi^g$ and $\Psi^g_0\in\mathcal{G}(Sp(n),\mathfrak{N}(n))$ (see Lemma 14 (2)). By applying Proposition 19 to $\Psi^g$ and $\Psi^g_0$, we have

$$\boldsymbol{K}_{\Psi^g}(H^i_n)=\boldsymbol{K}_0(H^i_n)=\boldsymbol{K}_{\Psi^g_0}(H^i_n);$$
$$\boldsymbol{K}_{\Psi^g}(H^i_{n-1}+H^i_n)=\boldsymbol{K}_0(H^i_{n-1}+H^i_n)=\boldsymbol{K}_{\Psi^g_0}(H^i_{n-1}+H^i_n)$$

for any $i\,(=1,2,3)$. Now assume that $X\in\mathcal{S}$ and that $g$ is an element of $Sp(n)$ such that $\mathrm{Ad}(g)X\in\mathbb{R}H^i_n$ or $\mathrm{Ad}(g)X\in\mathbb{R}(H^i_{n-1}+H^i_n)$. Then by the above equalities we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \boldsymbol{K}_{\Psi_0^g}(\mathrm{Ad}(g)X)$. (Note that $\boldsymbol{K}_{\Psi}(cZ) = \boldsymbol{K}_{\Psi}(Z)$ holds for any $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$, $Z \in \mathfrak{sp}(n)$ and $c \in \mathbb{R}\,(c \neq 0)$.) On account of Lemma 14 (1) we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_{\Psi}(X)$ and $\boldsymbol{K}_{\Psi_0^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_{\Psi_0}(X) = \mathrm{Ad}(g)\boldsymbol{K}_0(X)$. Therefore $\boldsymbol{K}_{\Psi}(X) = \boldsymbol{K}_0(X)$ follows immediately. □

As a consequence of Proposition 20 we can show

**Proposition 21.** *Let $i = 1$, $2$ or $3$. Then*

(1) $H_a^i \in \mathcal{S} \quad (1 \leq a \leq n)$;
(2) $H_a^i \pm H_b^i \in \mathcal{S} \quad (1 \leq a < b \leq n)$;
(3) $P_{ab} \in \mathcal{S},\ Q_{ab}^i \in \mathcal{S} \quad (1 \leq a < b \leq n)$.

*Consequently, for any $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ the following equalities hold:*

$$\begin{aligned} &\boldsymbol{K}_{\Psi}(H_a^i) = \boldsymbol{K}_0(H_a^i); \quad \boldsymbol{K}_{\Psi}(H_a^i \pm H_b^i) = \boldsymbol{K}_0(H_a^i \pm H_b^i); \\ &\boldsymbol{K}_{\Psi}(P_{ab}) = \boldsymbol{K}_0(P_{ab}); \quad \boldsymbol{K}_{\Psi}(Q_{ab}^i) = \boldsymbol{K}_0(Q_{ab}^i). \end{aligned} \tag{3.5}$$

**Proof.** Let $i = 1$, $2$ or $3$. It is easily shown that under the action of $Sp(n)$, $H_a^i\,(1 \leq a \leq n-1)$ is conjugate to $H_n^i$. This implies that $H_a^i \in \mathcal{S}\,(1 \leq a \leq n)$. It is also known that $H_a^i + H_b^i\,(1 \leq a < b \leq n)$ (resp. $H_a^i - H_b^i\,(1 \leq a < b \leq n)$) is conjugate to $H_{n-1}^i + H_n^i$ (resp. $H_{n-1}^i - H_n^i$). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then we easily have $[H_n^i, H_n^j] = 2\varepsilon(ijk)H_n^k$. This proves that $\mathfrak{s} = \sum_{i=1}^3 \mathbb{R}H_n^i$ is $NAT$. In view of the proof of Lemma 18 $\exp(\mathbb{R}H_n^k)$ acts on $\mathfrak{s}' = \mathbb{R}H_n^i + \mathbb{R}H_n^j$ as a non-trivial subgroup of rotations of $\mathfrak{s}'$. Hence, we can find an element $h \in \exp(\mathbb{R}H_n^k)$ such that $\mathrm{Ad}(h)H_n^i = -H_n^i$. Since $[H_n^k, H_{n-1}^i] = 0$, we have $\mathrm{Ad}(h)H_{n-1}^i = H_{n-1}^i$ and hence $\mathrm{Ad}(h)(H_{n-1}^i - H_n^i) = H_{n-1}^i + H_n^i$. Therefore, we have $H_a^i \pm H_b^i \in \mathcal{S}\,(1 \leq a < b \leq n)$. As we have pointed out, $P_{ab} \sim Q_{ab}^i \sim (H_a^i - H_b^i)$. Since $H_a^i - H_b^i \in \mathcal{S}$, it follows that $P_{ab} \in \mathcal{S}$ and $Q_{ab}^i \in \mathcal{S}$. This completes the proof. □

**Remark 22.** In the next section, after the proof of Theorem 10 we will know that $\boldsymbol{K}_{\Psi}(X) = \boldsymbol{K}_0(X)$ holds for any $X \in \mathfrak{sp}(n)$ (see Remark 36).

## 4. Solutions of the Gauss equation

In this section we will prove Theorem 10. We assume that $n \geq 2$ and that the Gaussian variety $\mathcal{G}(Sp(n'), \mathfrak{N}(n'))$ is $EOS$ for any $n'$ such that $n' < n$.

We now regard $\mathfrak{N}(n-1)$ as a subspace of $\mathfrak{N}(n)$ by the assignment

$$\mathfrak{N}(n-1) \ni Z \longmapsto \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{N}(n).$$

Let $\mathfrak{M}$ be the orthogonal complement of $\mathfrak{N}(n-1)$ in $\mathfrak{N}(n)$. Then we easily have $\dim \mathfrak{M} = 4n - 3$ and

$$\mathfrak{M} = \mathbb{R}E_{nn} + \sum_{a=1}^{n-1} \Big\{ \mathbb{R}(E_{an} + E_{na}) + \sum_{j=1}^{3} \mathbb{R}(E_{an} - E_{na})e^j \Big\} \quad \text{(orthogonal direct sum)}.$$

As in the previous section, we denote by $\boldsymbol{\Psi}_0$ the canonical solution (2.4). By a simple calculation we can easily verify that $\boldsymbol{\Psi}_0(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$ and $\mathfrak{M} = (\boldsymbol{\Psi}_0)_{H_n^i}(\mathfrak{sp}(n))\,(i = 1,2,3)$. In a natural manner, the restriction $\boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$ of $\boldsymbol{\Psi}_0$ to $\mathfrak{sp}(n-1)$ may be regarded as an element $\mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$. Therefore, by the hypothesis of our induction we have:

**Lemma 23.** *For any $\boldsymbol{\Psi}' \in \mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$ there is an element $\rho' \in O(\mathfrak{N}(n-1))$ such that $\rho'\boldsymbol{\Psi}' = \boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$.*

Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. By $\boldsymbol{V}_{\boldsymbol{\Psi}}(X)\ (\subset \mathfrak{N}(n))$ we denote the image of $\mathfrak{sp}(n)$ by the map $\boldsymbol{\Psi}_X$. We call $\boldsymbol{\Psi}$ a *normal solution* if $\boldsymbol{\Psi}$ satisfies:

(1) $\boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i) = \mathfrak{M}\,(i = 1,2,3)$;
(2) $\boldsymbol{\Psi}|_{\mathfrak{sp}(n-1)} = \boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$,

where $\boldsymbol{\Psi}|_{\mathfrak{sp}(n-1)}$ means the restriction of $\boldsymbol{\Psi}$ to $\mathfrak{sp}(n-1)$. By $\mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ we mean the subset of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$ consisting of all normal solutions.

**Proposition 24.** *Let $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\rho\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$.*

**Proof.** Since $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n^i) = \dim \boldsymbol{K}_0(H_n^i)$ (see Proposition 19), we have $\dim \boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i) = \dim \boldsymbol{V}_{\boldsymbol{\Psi}_0}(H_n^i)$. Hence we have $\dim \boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i) = \dim \mathfrak{M}$ for any $i\,(= 1,2,3)$. Let $X$, $Y \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we get

$$\frac{1}{4}\big([[X, H_n^i], Y], Z\big) = \big\langle \boldsymbol{\Psi}(X,Y), \boldsymbol{\Psi}(H_n^i, Z)\big\rangle - \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(H_n^i, Y)\big\rangle$$

for any $Z \in \mathfrak{sp}(n)$ and $i\,(= 1,2,3)$. Since $[X, H_n^i] = 0$ and $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n^i) = \boldsymbol{K}_0(H_n^i) \supset \mathfrak{sp}(n-1)$ (see (3.4) and Proposition 19), we have $\boldsymbol{\Psi}(H_n^i, Y) = 0$. Consequently, we have $\big\langle \boldsymbol{\Psi}(X,Y), \boldsymbol{\Psi}(H_n^i, Z)\big\rangle = 0$, which proves

$$\big\langle \boldsymbol{\Psi}(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)), \boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i)\big\rangle = 0. \tag{4.1}$$

Take an element $\rho_1 \in O(\mathfrak{N}(n))$ such that $\rho_1(\boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^1)) = \mathfrak{M}$. Then by (4.1) we have $(\rho_1\boldsymbol{\Psi})(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \rho_1(\boldsymbol{\Psi}(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1))) \subset \mathfrak{N}(n-1)$. Hence, in a natural manner, $(\rho_1\boldsymbol{\Psi})|_{\mathfrak{sp}(n-1)}$ may be regarded as an element of $\mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$. Hence there is an element $\rho_2' \in O(\mathfrak{N}(n-1))$ such that $\rho_2'((\rho_1\boldsymbol{\Psi})|_{\mathfrak{sp}(n-1)}) = \boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$ (see Lemma 23). Take $\rho_2 \in O(\mathfrak{N}(n))$ such that $\rho_2|_{\mathfrak{M}} = \mathbf{1}_{\mathfrak{M}}$ and $\rho_2|_{\mathfrak{N}(n-1)} = \rho_2'$. Put $\rho = \rho_2\rho_1$. Then we have $\boldsymbol{V}_{\rho\boldsymbol{\Psi}}(H_n^1) = \rho(\boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^1)) = \mathfrak{M}$ and $(\rho\boldsymbol{\Psi})|_{\mathfrak{sp}(n-1)} = \boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$. We finally prove $\boldsymbol{V}_{\rho\boldsymbol{\Psi}}(H_n^i) = \mathfrak{M}\,(i = 2,3)$. As is easily seen, we have $\boldsymbol{\Psi}(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \rho^{-1}(\mathfrak{N}(n-1))$. Hence by (4.1) we have $\boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i) \subset \rho^{-1}(\mathfrak{M})$. Therefore, $\boldsymbol{V}_{\rho\boldsymbol{\Psi}}(H_n^i) = \rho(\boldsymbol{V}_{\boldsymbol{\Psi}}(H_n^i)) \subset \mathfrak{M}$. Since $\dim \boldsymbol{V}_{\rho\boldsymbol{\Psi}}(H_n^i) = \dim \mathfrak{M}$, we have $\boldsymbol{V}_{\rho\boldsymbol{\Psi}}(H_n^i) = \mathfrak{M}$, implying $\rho\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. This completes the proof. □

By virtue of Proposition 24 to show Theorem 10 it suffices to prove that any element of $\mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ is equivalent to $\boldsymbol{\Psi}_0$.

By $\mathfrak{m}$ we denote the orthogonal complement of $\mathfrak{sp}(n-1)$ in $\mathfrak{sp}(n)$. For simplicity, we set $P_a = P_{an}$, $Q^i_a = Q^i_{an}$ and $H^i = H^i_n$ for integers $a\,(1 \le a \le n-1)$ and $i\,(1 \le i \le 3)$. Set

$$\mathfrak{m}_a = \mathbb{R}P_a + \sum_{i=1}^{3} \mathbb{R}Q^i_a \ (1 \le a \le n-1), \qquad \mathfrak{m}_n = \sum_{i=1}^{3} \mathbb{R}H^i.$$

Since $\big(\mathfrak{m}_a, \mathfrak{m}_b\big) = 0\,(a \ne b)$, we have

$$\mathfrak{m} = \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n \quad \text{(orthogonal direct sum)}.$$

**Lemma 25.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ *and let* $i = 1,\ 2$ *or* $3$. *Then*:

$$\mathfrak{M} = \sum_{a=1}^{n-1} \boldsymbol{\Psi}(H^i, \mathfrak{m}_a) + \mathbb{R}\boldsymbol{\Psi}(H^i, H^i) \quad (\textit{direct sum}).$$

**Proof.** Since $\boldsymbol{K}_{\boldsymbol{\Psi}}(H^i) = \mathfrak{sp}(n-1) + \sum_{j \ne i} \mathbb{R}H^j$ and $\boldsymbol{V}_{\boldsymbol{\Psi}}(H^i) = \boldsymbol{\Psi}(H^i, \mathfrak{m}) = \mathfrak{M}$, we have the lemma. □

In what follows we will observe the value $\boldsymbol{\Psi}(X, Y)\ (X, Y \in \mathfrak{sp}(n))$ for the following four cases:

(I) $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
(II) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_n$;
(III) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a\ (1 \le a \le n-1)$;
(IV) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_a\ (1 \le a \le n-1)$.

We first observe Case (I):

**Proposition 26.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Then*:

(1) $\boldsymbol{\Psi}(\mathfrak{m}, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$.
(2) *Let* $X$, $Y \in \mathfrak{m}$ *and* $Z \in \mathfrak{sp}(n-1)$. *Then*:

$$\big\langle \boldsymbol{\Psi}(X, Z), \boldsymbol{\Psi}(H^i, Y) \big\rangle = \frac{1}{4}\big([[X, Z], H^i], Y\big). \tag{4.2}$$

**Proof.** We first note that $\boldsymbol{\Psi}(H^i, \mathfrak{sp}(n-1)) = 0\,(1 \le i \le 3)$, because $\boldsymbol{K}_{\boldsymbol{\Psi}}(H^i) \supset \mathfrak{sp}(n-1)$. This proves $\boldsymbol{\Psi}(\mathfrak{m}_n, \mathfrak{sp}(n-1)) = 0$. We now prove $\boldsymbol{\Psi}(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$ for any $a\,(1 \le a \le n-1)$. To show this we prove

$$\boldsymbol{\Psi}(P_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}; \qquad \boldsymbol{\Psi}(Q^i_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M} \quad (i = 1, 2, 3). \tag{4.3}$$

Define an element $Z^i_0 \in \mathfrak{sp}(n-1)\ (1 \le i \le 3)$ by $Z^i_0 = (\sum_{s=1}^{n-1} sE_{ss})e^i$. Then it is well-known that $Z^i_0$ is a regular element of $\mathfrak{sp}(n-1)$. Moreover, since $\boldsymbol{\Psi}|_{\mathfrak{sp}(n-1)} = \boldsymbol{\Psi}_0|_{\mathfrak{sp}(n-1)}$, it follows that $\boldsymbol{\Psi}(Z^i_0, \mathfrak{sp}(n-1)) \subset \mathfrak{N}(n-1)$. Here we note that the equality $\boldsymbol{\Psi}(Z^i_0, \mathfrak{sp}(n-1)) =$

$\mathfrak{N}(n-1)$ holds. Indeed, since $\dim \mathbf{Ker}((\boldsymbol{\Psi}_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) = 2(n-1)$ (see Proposition 15), we have

$$\dim \boldsymbol{\Psi}(Z_0^i, \mathfrak{sp}(n-1)) = \dim \mathfrak{sp}(n-1) - \dim \mathbf{Ker}((\boldsymbol{\Psi}_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) = \dim \mathfrak{N}(n-1).$$

Now let us set $W_a^i = Z_0^i - aH^i \in \mathfrak{sp}(n)$ $(1 \le a \le n-1)$. By a direct calculation we can verify $\boldsymbol{\Psi}_0(P_a, W_a^i) = \boldsymbol{\Psi}_0(Q_a^i, W_a^i) = 0$. Hence by (3.5) we have $\boldsymbol{\Psi}(P_a, W_a^i) = \boldsymbol{\Psi}(Q_a^i, W_a^i) = 0$. Moreover, since $\boldsymbol{\Psi}(H^i, \mathfrak{sp}(n-1)) = 0$, we have $\boldsymbol{\Psi}(W_a^i, \mathfrak{sp}(n-1)) = \boldsymbol{\Psi}(Z_0^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$. Let $Z$, $Z' \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[W_a^i, Z], Z'], P_a\big) = \big\langle \boldsymbol{\Psi}(W_a^i, Z'), \boldsymbol{\Psi}(Z, P_a)\big\rangle - \big\langle \boldsymbol{\Psi}(W_a^i, P_a), \boldsymbol{\Psi}(Z, Z')\big\rangle, \tag{4.4}$$

$$\frac{1}{4}\big([[W_a^i, Z], Z'], Q_a^i\big) = \big\langle \boldsymbol{\Psi}(W_a^i, Z'), \boldsymbol{\Psi}(Z, Q_a^i)\big\rangle - \big\langle \boldsymbol{\Psi}(W_a^i, Q_a^i), \boldsymbol{\Psi}(Z, Z')\big\rangle. \tag{4.5}$$

Since $[H^i, Z] = 0$, we have $[[W_a^i, Z], Z'] = [[Z_0^i, Z], Z'] \in \mathfrak{sp}(n-1)$. Hence, the left sides of (4.4) and (4.5) vanish. Further, since $\boldsymbol{\Psi}(P_a, W_a^i) = \boldsymbol{\Psi}(Q_a^i, W_a^i) = 0$, we have $\big\langle \boldsymbol{\Psi}(W_a^i, Z'), \boldsymbol{\Psi}(Z, P_a)\big\rangle = \big\langle \boldsymbol{\Psi}(W_a^i, Z'), \boldsymbol{\Psi}(Z, Q_a^i)\big\rangle = 0$. Since $Z$ and $Z'$ are arbitrary elements of $\mathfrak{sp}(n-1)$ and since $\boldsymbol{\Psi}(W_a^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$, we have

$$\big\langle \mathfrak{N}(n-1), \boldsymbol{\Psi}(\mathfrak{sp}(n-1), P_a)\big\rangle = \big\langle \mathfrak{N}(n-1), \boldsymbol{\Psi}(\mathfrak{sp}(n-1), Q_a^i)\big\rangle = 0,$$

showing (4.3). Consequently, we have $\boldsymbol{\Psi}(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$, which completes the proof of (1).

Next we show (2). Let $X$, $Y \in \mathfrak{m}$ and $Z \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[X, H^i], Z], Y\big) = \big\langle \boldsymbol{\Psi}(X, Z), \boldsymbol{\Psi}(H^i, Y)\big\rangle - \big\langle \boldsymbol{\Psi}(X, Y), \boldsymbol{\Psi}(H^i, Z)\big\rangle.$$

Note that $\boldsymbol{\Psi}(H^i, Z) = 0$ and $[Z, H^i] = 0$. The latter equality, together with the Jacobi identity, shows $[[X, H^i], Z] = [[X, Z], H^i]$. Thus we obtain (4.2). ☐

**Remark 27.** Here we state a remark on the value $\boldsymbol{\Psi}(X, Z)$ $(X \in \mathfrak{m}$, $Z \in \mathfrak{sp}(n-1))$. Note that the right side of (4.2) is an intrinsic quantity. Since $\boldsymbol{\Psi}(H^i, \mathfrak{m}) = \mathfrak{M}$, we know that $\boldsymbol{\Psi}(X, Z) \in \mathfrak{M}$ is uniquely determined if the values $\boldsymbol{\Psi}(H^i, Y)$ $(Y \in \mathfrak{m})$ are given. Therefore, if $\boldsymbol{\Psi}(H^i, Y) = \boldsymbol{\Psi}_0(H^i, Y)$ holds for any $Y \in \mathfrak{m}$, then we may conclude that $\boldsymbol{\Psi}(X, Z) = \boldsymbol{\Psi}_0(X, Z)$ $(X \in \mathfrak{m}, Z \in \mathfrak{sp}(n-1))$. See Case $(c)$ below in the proof of Theorem 10.

We next observe Case (II):

**Proposition 28.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Then*:

(1) $\boldsymbol{\Psi}(H^1, H^1) = \boldsymbol{\Psi}(H^2, H^2) = \boldsymbol{\Psi}(H^3, H^3)$.
(2) $\boldsymbol{\Psi}(H^1, H^2) = \boldsymbol{\Psi}(H^2, H^3) = \boldsymbol{\Psi}(H^3, H^1) = 0$.
(3) $\big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(H^i, H^i)\big\rangle = 1 \quad (1 \le i \le 3)$.
(4) $\big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(H^i, \mathfrak{m}_a)\big\rangle = 0 \quad (1 \le i \le 3,\ 1 \le a \le n-1)$.

To prove the proposition we prepare

**Lemma 29.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. *Let* $X$ *and* $Y \in \mathfrak{sp}(n)$. *Assume*:

(i) $\boldsymbol{\Psi}_0(X, X) = \boldsymbol{\Psi}_0(Y, Y)$.
(ii) $X + Y \in \mathcal{S}$.

*Then* $\boldsymbol{\Psi}(X, X) = \boldsymbol{\Psi}(Y, Y)$.

**Proof.** By (i) we easily have $\boldsymbol{\Psi}_0(X+Y, X-Y) = 0$, i.e., $X - Y \in \boldsymbol{K}_0(X+Y)$. Since $X + Y \in \mathcal{S}$, we have $\boldsymbol{K}_0(X+Y) = \boldsymbol{K}_{\boldsymbol{\Psi}}(X+Y)$ (see Proposition 20). Consequently, it follows that $X - Y \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X+Y)$, i.e., $\boldsymbol{\Psi}(X+Y, X-Y) = 0$. This implies $\boldsymbol{\Psi}(X, X) = \boldsymbol{\Psi}(Y, Y)$. □

**Proof of Proposition** 28. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. As shown in the proof of Proposition 21, $\mathfrak{s} = \sum_{i=1}^{3} \mathbb{R}H^i$ is $NAT$. Consequently, $H^i + H^j \in \mathcal{S}$, because $(H^i + H^j) \sim H^i$. On the other hand, it is easily checked that $\boldsymbol{\Psi}_0(H^i, H^i) = \boldsymbol{\Psi}_0(H^j, H^j) = -E_{nn}$. Hence by Lemma 29 we have $\boldsymbol{\Psi}(H^i, H^i) = \boldsymbol{\Psi}(H^j, H^j)$. Similarly, we have $\boldsymbol{\Psi}(H^j, H^j) = \boldsymbol{\Psi}(H^k, H^k)$, proving (1). The assertion (2) is clear from Lemma 13. Finally we prove (3) and (4). Let $k$ be an integer such that $1 \leq k \leq 3$, $k \neq i$ and $X \in \mathfrak{sp}(n)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[H^i, H^k], H^k], X\big) = \big\langle \boldsymbol{\Psi}(H^i, H^k), \boldsymbol{\Psi}(H^k, X)\big\rangle - \big\langle \boldsymbol{\Psi}(H^i, X), \boldsymbol{\Psi}(H^k, H^k)\big\rangle.$$

By a simple calculation we have $[[H^i, H^k], H^k] = -4H^i$. Moreover, by the results obtained in (1) and (2) we have $\boldsymbol{\Psi}(H^i, H^k) = 0$ and $\boldsymbol{\Psi}(H^k, H^k) = \boldsymbol{\Psi}(H^i, H^i)$. Consequently, we have

$$\big\langle \boldsymbol{\Psi}(H^i, X), \boldsymbol{\Psi}(H^i, H^i)\big\rangle = \big(H^i, X\big).$$

Therefore, we obtain (3) and (4), because $\big(H^i, H^i\big) = 1$ and $\big(H^i, \mathfrak{m}_a\big) = 0$ (see (3.3)). □

In Case (III) the value $\boldsymbol{\Psi}(X, Y)$ $(X, Y \in \mathfrak{m}_a)$ $(1 \leq a \leq n-1)$ are determined by

**Proposition 30.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ *and let* $a$ *be an integer such that* $1 \leq a \leq n-1$. *Then*:

(1) $\boldsymbol{\Psi}(P_a, Q_a^i) = 0 \quad (1 \leq i \leq 3)$.
(2) $\boldsymbol{\Psi}(Q_a^i, Q_a^j) = 0 \quad (1 \leq i \neq j \leq 3)$.
(3) $\boldsymbol{\Psi}(P_a, P_a) = \boldsymbol{\Psi}(Q_a^i, Q_a^i) = \boldsymbol{\Psi}(H^i, H^i) + \boldsymbol{\Psi}(H_a^i, H_a^i) \quad (1 \leq i \leq 3)$.

**Proof.** Since $\boldsymbol{\Psi}_0(P_a, Q_a^i) = 0$ and $\boldsymbol{\Psi}_0(Q_a^i, Q_a^j) = 0$ $(i \neq j)$, we obtain (1) and (2) (see (3.5)). We now prove (3). Since $\mathfrak{s}_{an}^i = \mathbb{R}(H^i - H_a^i) + \mathbb{R}P_a + \mathbb{R}Q_a^i$ is $NAT$, it follows that $Q_a^i + (H^i - H_a^i) \in \mathcal{S}$. Indeed, $Q_a^i + (H^i - H_a^i) \sim (H^i - H_a^i)$. By Lemma 29 we have $\boldsymbol{\Psi}(Q_a^i, Q_a^i) = \boldsymbol{\Psi}(H^i - H_a^i, H^i - H_a^i)$, because $\boldsymbol{\Psi}_0(Q_a^i, Q_a^i) = \boldsymbol{\Psi}_0(H^i - H_a^i, H^i - H_a^i) = -(E_{aa} + E_{nn})$. Since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\boldsymbol{\Psi}(H^i, H_a^i) = 0$. Consequently, $\boldsymbol{\Psi}(Q_a^i, Q_a^i) = \boldsymbol{\Psi}(H^i, H^i) + \boldsymbol{\Psi}(H_a^i, H_a^i)$. Similarly, we can prove $\boldsymbol{\Psi}(P_a, P_a) = \boldsymbol{\Psi}(H^i, H^i) + \boldsymbol{\Psi}(H_a^i, H_a^i)$. □

Before proceeding to Case (IV) we extend Lemma 29 to the following form:

**Lemma 31.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. *Let* $X$, $X'$, $Y$ *and* $Y' \in \mathfrak{sp}(n)$. *Assume*:

(i) $\boldsymbol{\Psi}_0(X, Y') = \boldsymbol{\Psi}_0(Y, X') = 0$.
(ii) $\boldsymbol{\Psi}_0(X, X') = \boldsymbol{\Psi}_0(Y, Y')$.
(iii) $X \in \mathcal{S}$, $Y \in \mathcal{S}$ *and* $X + Y \in \mathcal{S}$.

*Then* $\boldsymbol{\Psi}(X, X') = \boldsymbol{\Psi}(Y, Y')$.

**Proof.** By (i) and (ii) we have $Y' \in \boldsymbol{K}_0(X)$, $X' \in \boldsymbol{K}_0(Y)$ and $\boldsymbol{\Psi}_0(X+Y, X'-Y') = 0$. The last equality implies that $X' - Y' \in \boldsymbol{K}_0(X+Y)$. Hence by (iii) we have $Y' \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X)$, $X' \in \boldsymbol{K}_{\boldsymbol{\Psi}}(Y)$ and $X' - Y' \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X+Y)$. Consequently, we have $\boldsymbol{\Psi}(Y', X) = \boldsymbol{\Psi}(X', Y) = \boldsymbol{\Psi}(X+Y, X'-Y') = 0$. Hence $\boldsymbol{\Psi}(X, X') = \boldsymbol{\Psi}(Y, Y')$. □

With this preparation we observe Case (IV).

**Proposition 32.** *Let* $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Let* $a$ *be an integer such that* $1 \leq a \leq n-1$. *Then*:

(1) $\boldsymbol{\Psi}(H^1, Q_a^1) = \boldsymbol{\Psi}(H^2, Q_a^2) = \boldsymbol{\Psi}(H^3, Q_a^3)$.
(2) $\boldsymbol{\Psi}(H^i, Q_a^j) = -\varepsilon(ijk)\boldsymbol{\Psi}(H^k, P_a)$, *where* $\{i, j, k\}$ *is a permutation of* $\{1, 2, 3\}$.
(3) $\boldsymbol{\Psi}(H^1, \mathfrak{m}_a) = \boldsymbol{\Psi}(H^2, \mathfrak{m}_a) = \boldsymbol{\Psi}(H^3, \mathfrak{m}_a)$.
(4) *For each* $i$ $(1 \leq i \leq 3)$ *the set* $\{\sqrt{2}\boldsymbol{\Psi}(H^i, P_a), \sqrt{2}\boldsymbol{\Psi}(H^i, Q_a^j)\ (1 \leq j \leq 3)\}$ *forms an orthonormal basis of* $\boldsymbol{\Psi}(H^i, \mathfrak{m}_a)$.

**Proof.** Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. We note that the subspace $\mathfrak{s} = \mathbb{R}(H_a^i + H^i) + \mathbb{R}Q_a^j + \mathbb{R}Q_a^k$ forms a subalgebra of $\mathfrak{sp}(n)$ and is $NAT$. In fact, by simple calculations we have

$$[H_a^i + H^i, Q_a^j] = 2\varepsilon(ijk)Q_a^k; \quad [H_a^i + H^i, Q_a^k] = -2\varepsilon(ijk)Q_a^j;$$
$$[Q_a^j, Q_a^k] = 2\varepsilon(ijk)(H_a^i + H^i).$$

Hence we have $H_a^i + H^i + Q_a^j \in \mathcal{S}$ and $H_a^i + H^i + Q_a^k \in \mathcal{S}$, because $H_a^i + H^i + Q_a^j \sim H_a^i + H^i + Q_a^k \sim H_a^i + H^i \in \mathcal{S}$.

Now we prove (1). By direct calculations we can show $\boldsymbol{\Psi}_0(H_a^1 + H^1, Q_a^1) = \boldsymbol{\Psi}_0(H_a^2 + H^2, Q_a^2) = \boldsymbol{\Psi}_0(H_a^3 + H^3, Q_a^3) = -(E_{an} + E_{na})$. Moreover we have $\boldsymbol{\Psi}_0(H_a^i + H^i, H_a^j + H^j) = \boldsymbol{\Psi}_0(Q_a^i, Q_a^j) = 0$ if $i \neq j$ (see Lemma 13 and Proposition 30). Therefore by Lemma 31 we have

$$\boldsymbol{\Psi}(H_a^1 + H^1, Q_a^1) = \boldsymbol{\Psi}(H_a^2 + H^2, Q_a^2) = \boldsymbol{\Psi}(H_a^3 + H^3, Q_a^3). \tag{4.6}$$

Here we show $\boldsymbol{\Psi}(H_a^1, Q_a^1) = \boldsymbol{\Psi}(H_a^2, Q_a^2) = \boldsymbol{\Psi}(H_a^3, Q_a^3)$. Let $i = 1$, 2 or 3. Since $H_a^i \in \mathfrak{sp}(n-1)$ and $Q_a^i \in \mathfrak{m}$, it follows from Proposition 26 (1) that $\boldsymbol{\Psi}(H_a^i, Q_a^i) \in \mathfrak{M}$. Moreover, by Proposition 26 (2) we have

$$\big\langle \boldsymbol{\Psi}(Q_a^i, H_a^i), \boldsymbol{\Psi}(H^1, Y) \big\rangle = \frac{1}{4}\big([[Q_a^i, H_a^i], H^1], Y\big)$$

for any $Y \in \mathfrak{m}$. Since $[Q^i_a, H^i_a] = P_a$, the right side of the above equality does not depend on the choice of $i$. This implies that $\boldsymbol{\Psi}(H^1_a, Q^1_a) = \boldsymbol{\Psi}(H^2_a, Q^2_a) = \boldsymbol{\Psi}(H^3_a, Q^3_a)$, because $\boldsymbol{\Psi}(H^1, \mathfrak{m}) = \mathfrak{M}$. This, together with (4.6), proves (1).

We next prove (2). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then by direct calculations we have $\boldsymbol{\Psi}_0(H^i_a - H^i, Q^j_a) = \varepsilon(ijk)\boldsymbol{\Psi}_0(H^k_a + H^k, P_a) = \varepsilon(ijk)(E_{an} - E_{na})e^k$. Moreover, $\boldsymbol{\Psi}_0(H^i_a - H^i, H^k_a + H^k) = \boldsymbol{\Psi}_0(Q^j_a, P_a) = 0$ (see Lemma 13 and Proposition 30). Since $H^k_a + H^k + Q^j_a \in \mathcal{S}$, we obtain by Lemma 31 the following

$$\boldsymbol{\Psi}(H^i_a - H^i, Q^j_a) = \varepsilon(ijk)\boldsymbol{\Psi}(H^k_a + H^k, P_a). \tag{4.7}$$

Note that $H^i_a, H^k_a \in \mathfrak{sp}(n-1)$, $Q^j_a, P_a \in \mathfrak{m}$ and $[Q^j_a, H^i_a] = \varepsilon(ijk)[P_a, H^k_a] = -\varepsilon(ijk)Q^k_a$. As in the proof of (1) we have $\boldsymbol{\Psi}(H^i_a, Q^j_a) = \varepsilon(ijk)\boldsymbol{\Psi}(H^k_a, P_a)$. Accordingly, from (4.7) we have $\boldsymbol{\Psi}(H^i, Q^j_a) = -\varepsilon(ijk)\boldsymbol{\Psi}(H^k, P_a)$. This completes the proof of (2).

By (1) and (2) we have

$$\begin{aligned}
\boldsymbol{\Psi}(H^1, P_a) &= -\boldsymbol{\Psi}(H^2, Q^3_a) = \boldsymbol{\Psi}(H^3, Q^2_a);\\
\boldsymbol{\Psi}(H^1, Q^1_a) &= \boldsymbol{\Psi}(H^2, Q^2_a) = \boldsymbol{\Psi}(H^3, Q^3_a);\\
\boldsymbol{\Psi}(H^1, Q^2_a) &= -\boldsymbol{\Psi}(H^2, Q^1_a) = -\boldsymbol{\Psi}(H^3, P_a);\\
\boldsymbol{\Psi}(H^1, Q^3_a) &= \boldsymbol{\Psi}(H^2, P_a) = -\boldsymbol{\Psi}(H^3, Q^1_a).
\end{aligned} \tag{4.8}$$

By these equalities we clearly obtain (3).

Finally, we prove (4). Let $X$ and $Y$ are one of $P_a$ and $Q^j_a$ $(1 \le j \le 3)$, i.e., $X$, $Y \in \{P_a, Q^j_a\ (1 \le j \le 3)\}$. By the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[H^i, X], H^i], Y\big) = \big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(X, Y)\big\rangle - \big\langle \boldsymbol{\Psi}(H^i, Y), \boldsymbol{\Psi}(X, H^i)\big\rangle.$$

By direct calculations we can verify $[[H^i, X], H^i] = X$. Hence the left side of the above equality becomes $(1/4)\big(X, Y\big)$. First assume that $X = Y$. Then we have $\boldsymbol{\Psi}(X, X) = \boldsymbol{\Psi}(H^i, H^i) + \boldsymbol{\Psi}(H^i_a, H^i_a)$ (see Proposition 30 (3)). Since $\big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(H^i, H^i)\big\rangle = 1$ (see Proposition 28), $\boldsymbol{\Psi}(H^i, H^i) \in \mathfrak{M}$ and $\boldsymbol{\Psi}(H^i_a, H^i_a) \in \mathfrak{N}(n-1)$, we have

$$\big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(X, X)\big\rangle = \big\langle \boldsymbol{\Psi}(H^i, H^i), \boldsymbol{\Psi}(H^i, H^i) + \boldsymbol{\Psi}(H^i_a, H^i_a)\big\rangle = 1.$$

Since $\big(X, X\big) = 2$ (see (3.3)), we have $\big\langle \boldsymbol{\Psi}(H^i, X), \boldsymbol{\Psi}(H^i, X)\big\rangle = 1/2$. We next consider the case $X \neq Y$. Then we have $\big(X, Y\big) = 0$ and $\boldsymbol{\Psi}(X, Y) = 0$ (see (3.3) and Proposition 30 (1), (2)). Hence it follows that $\big\langle \boldsymbol{\Psi}(H^i, X), \boldsymbol{\Psi}(H^i, Y)\big\rangle = 0$. This completes the proof of (4). □

We are now in a position to prove Theorem 10.

**Proof of Theorem 10.** Let $\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. Set $\boldsymbol{H} = \boldsymbol{\Psi}(H^1, H^1)$, $\boldsymbol{P}_a = \sqrt{2}\boldsymbol{\Psi}(H^1, P_a)$ $(1 \le a \le n-1)$, $\boldsymbol{Q}^i_a = \sqrt{2}\boldsymbol{\Psi}(H^1, Q^i_a)$ $(1 \le a \le n-1,\ 1 \le i \le 3)$. Then we have

**Lemma 33.** *The set* $\mathfrak{O} = \{\boldsymbol{H}, \boldsymbol{P}_a\,(1 \leq a \leq n-1), \boldsymbol{Q}_a^i\,(1 \leq a \leq n-1,\ 1 \leq i \leq 3)\}$ *forms an orthonormal basis of* $\mathfrak{M}$.

**Proof.** By virtue of Proposition 28 (3), (4) and Proposition 32 (4) we have only to prove

$$\big\langle \boldsymbol{\Psi}(H^1, \mathfrak{m}_a), \boldsymbol{\Psi}(H^1, \mathfrak{m}_b) \big\rangle = 0 \quad (1 \leq a \neq b \leq n-1). \tag{4.9}$$

Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. By the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[H^1, X], H^2], Y\big) = \big\langle \boldsymbol{\Psi}(H^1, H^2), \boldsymbol{\Psi}(X, Y)\big\rangle - \big\langle \boldsymbol{\Psi}(H^1, Y), \boldsymbol{\Psi}(X, H^2)\big\rangle.$$

As is easily seen, $[[H^1, X], H^2] \in \mathfrak{m}_a$. Hence the left side of the above equality vanishes. On the other hand, since $\boldsymbol{\Psi}(H^1, H^2) = 0$ (see Proposition 28), it follows that $\big\langle \boldsymbol{\Psi}(H^1, Y), \boldsymbol{\Psi}(X, H^2)\big\rangle = 0$. This proves that $\big\langle \boldsymbol{\Psi}(H^1, \mathfrak{m}_b), \boldsymbol{\Psi}(H^2, \mathfrak{m}_a)\big\rangle = 0$. Therefore, we obtain (4.9), because $\boldsymbol{\Psi}(H^2, \mathfrak{m}_a) = \boldsymbol{\Psi}(H^1, \mathfrak{m}_a)$ (see Proposition 32 (3)). This completes the proof. □

Let $\mathfrak{O}_0 = \{\boldsymbol{H}_0, (\boldsymbol{P}_a)_0\,(1 \leq a \leq n-1),\ (\boldsymbol{Q}_a^i)_0\,(1 \leq a \leq n-1,\ 1 \leq i \leq 3)\}$ be the orthonormal basis of $\mathfrak{M}$ corresponding to $\boldsymbol{\Psi}_0$, i.e., $\boldsymbol{H}_0 = \boldsymbol{\Psi}_0(H^1, H^1)$, $(\boldsymbol{P}_a)_0 = \sqrt{2}\boldsymbol{\Psi}_0(H^1, P_a)$ and $(\boldsymbol{Q}_a^i)_0 = \sqrt{2}\boldsymbol{\Psi}_0(H^1, Q_a^i)$. Then, there is an orthogonal transformation $\rho'$ of $\mathfrak{M}$ such that $\boldsymbol{H}_0 = \rho'(\boldsymbol{H})$, $(\boldsymbol{P}_a)_0 = \rho'(\boldsymbol{P}_a)$ and $(\boldsymbol{Q}_a^i)_0 = \rho'(\boldsymbol{Q}_a^i)$. Extend $\rho'$ to the orthogonal transformation $\rho$ of $\mathfrak{N}(n)$ satisfying $\rho|_{\mathfrak{M}} = \rho'$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$. Then, it is easy to see that $\rho\boldsymbol{\Psi} \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. For simplicity, set $\boldsymbol{\Psi}_1 = \rho\boldsymbol{\Psi}$. In the following we will prove $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_0$. In view of Lemma 25 and the decomposition $\mathfrak{sp}(n) = \mathfrak{m} + \mathfrak{sp}(n-1)$, we may conclude $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_0$ if $\boldsymbol{\Psi}_1(X, Y) = \boldsymbol{\Psi}_0(X, Y)$ holds for any pairs $X$ and $Y$ listed in the following $(a) \sim (e)$:

- $(a)$ $X \in \mathfrak{sp}(n-1)$ and $Y \in \mathfrak{sp}(n-1)$;
- $(b)$ $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}$;
- $(c)$ $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
- $(d)$ $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a\,(1 \leq a \leq n-1)$;
- $(e)$ $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b\,(1 \leq a \neq b \leq n-1)$.

**Case** $(a)$**.** Let $X, Y \in \mathfrak{sp}(n-1)$. Since $\boldsymbol{\Psi}(X, Y) = \boldsymbol{\Psi}_0(X, Y) \in \mathfrak{N}(n-1)$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$, we have $\boldsymbol{\Psi}_1(X, Y) = \rho(\boldsymbol{\Psi}(X, Y)) = \rho(\boldsymbol{\Psi}_0(X, Y)) = \boldsymbol{\Psi}_0(X, Y)$.

**Case** $(b)$**.** By the very definition of $\rho$ we have $\boldsymbol{\Psi}_1(H^1, Y) = \boldsymbol{\Psi}_0(H^1, Y)$ for $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathbb{R}H^1$. Applying Proposition 32 to both $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_0$, we have $\boldsymbol{\Psi}_1(H^i, Y) = \boldsymbol{\Psi}_0(H^i, Y)$ for $i = 2, 3$, $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a$ (see (1), (2) and (4.8)). Further, since $\boldsymbol{\Psi}_1(H^1, H^1) = \boldsymbol{\Psi}_0(H^1, H^1)$, we have $\boldsymbol{\Psi}_1(H^i, H^j) = \boldsymbol{\Psi}_0(H^i, H^j)\,(1 \leq i, j \leq 3)$ (see Proposition 28 (1), (2)). Thus we obtain $\boldsymbol{\Psi}_1(X, Y) = \boldsymbol{\Psi}_0(X, Y)$ for any $X \in \mathfrak{m}_n$ and $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n = \mathfrak{m}$.

**Case** $(c)$**.** By Case $(b)$ we have $\boldsymbol{\Psi}_1(H^i, Y) = \boldsymbol{\Psi}_0(H^i, Y)\,(i = 1, 2, 3; Y \in \mathfrak{m})$. As we have remarked (see Remark 27), we obtain $\boldsymbol{\Psi}_1(X, Y) = \boldsymbol{\Psi}_0(X, Y)$ for $X \in \mathfrak{m}$, $Y \in \mathfrak{sp}(n-1)$.

**Case** $(d)$**.** As seen in Case $(b)$, we have $\mathbf{\Psi}_1(H^i, H^i) = \mathbf{\Psi}_0(H^i, H^i)$. Moreover, since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\mathbf{\Psi}_1(H_a^i, H_a^i) = \mathbf{\Psi}_0(H_a^i, H_a^i)\,(i = 1, 2, 3)$. Hence by applying Proposition 30 to $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_0$, we easily have $\mathbf{\Psi}_1(X, Y) = \mathbf{\Psi}_0(X, Y)$ for $X$, $Y \in \mathfrak{m}_a$.

**Case** $(e)$**.** We note that this case occurs when $n \geq 3$. We first show

**Lemma 34.** *Assume that* $n \geq 3$*. Let* $a$ *and* $c$ *be integers such that* $1 \leq a \neq c \leq n-1$*. Then* $P_a \pm P_{ac} \in \mathcal{S}$; $Q_a^i \pm Q_{ac}^i \in \mathcal{S}\,(i = 1, 2, 3)$*.*

**Proof.** By easy calculations we have

$$[H_c^i - H^i, P_a \pm P_{ac}] = Q_a^i \mp Q_{ac}^i; \qquad [H_c^i - H^i, Q_a^i \mp Q_{ac}^i] = -(P_a \pm P_{ac});$$
$$[P_a \pm P_{ac}, Q_a^i \mp Q_{ac}^i] = 2(H_c^i - H^i).$$

Consequently, both the subspaces $\mathfrak{s}_+ = \mathbb{R}(H_c^i - H^i) + \mathbb{R}(P_a + P_{ac}) + \mathbb{R}(Q_a^i - Q_{ac}^i)$ and $\mathfrak{s}_- = \mathbb{R}(H_c^i - H^i) + \mathbb{R}(P_a - P_{ac}) + \mathbb{R}(Q_a^i + Q_{ac}^i)$ are $NAT$. Therefore, we have $P_a \pm P_{ac} \sim H_c^i - H^i \sim Q_a^i \pm Q_{ac}^i$. Since $H_c^i - H^i \in \mathcal{S}$, it follows that $P_a \pm P_{ac} \in \mathcal{S}$ and $Q_a^i \pm Q_{ac}^i \in \mathcal{S}$. □

First assume $n \geq 4$. Let us consider the case $X = P_a$ and $Y = P_b$. Take an integer $c\,(1 \leq c \leq n-1)$ such that $c \neq a$ and $c \neq b$. By easy calculations we have $\mathbf{\Psi}_0(P_a, P_b) = \mathbf{\Psi}_0(P_{ac}, P_{bc}) = -(1/2)(E_{ab} + E_{ba})$ and $\mathbf{\Psi}_0(P_a, P_{bc}) = \mathbf{\Psi}_0(P_{ac}, P_b) = 0$. Since $P_a$, $P_{ac}$ and $P_a + P_{ac} \in \mathcal{S}$, it follows that $\mathbf{\Psi}_1(P_a, P_b) = \mathbf{\Psi}_1(P_{ac}, P_{bc})$ (see Lemma 31). Since $P_{ac}$, $P_{bc} \in \mathfrak{sp}(n-1)$, we have $\mathbf{\Psi}_1(P_{ac}, P_{bc}) = \mathbf{\Psi}_0(P_{ac}, P_{bc})$ (see the case $(a)$). Hence we have $\mathbf{\Psi}_1(P_a, P_b) = \mathbf{\Psi}_0(P_a, P_b)$. In a similar manner we can prove $\mathbf{\Psi}_1(P_a, Q_b^i) = \mathbf{\Psi}_0(P_a, Q_b^i)\,(i = 1, 2, 3)$ and $\mathbf{\Psi}_1(Q_a^i, Q_b^j) = \mathbf{\Psi}_0(Q_a^i, Q_b^j)\,(i, j = 1, 2, 3)$. By these facts we obtain the equality $\mathbf{\Psi}_1(X, Y) = \mathbf{\Psi}_0(X, Y)\,(X \in \mathfrak{m}_a,\ Y \in \mathfrak{m}_b)$ when $n \geq 4$.

Next we assume $n = 3$. Apparently, the method used in the case $n \geq 4$ cannot be applied to this case. We prove

**Lemma 35.** *Assume that* $n = 3$*. Then* $\mathbf{\Psi}_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$*.*

**Proof.** Set $\mathfrak{B}_a = \{P_a, Q_a^1, Q_a^2, Q_a^3\}\,(a = 1, 2)$. Let $X \in \mathfrak{B}_1$ and $Y \in \mathfrak{B}_2$. We first show

$$\big\langle \mathbf{\Psi}_1(X, Y), \mathbf{\Psi}_1(H^1, H^1) \big\rangle = \big\langle \mathbf{\Psi}_1(X, Y), \mathbf{\Psi}_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2) \big\rangle = 0. \tag{4.10}$$

If this is true, then we have $\mathbf{\Psi}_1(X, Y) \in \mathfrak{N}(2)$, because $\mathfrak{M} = \mathbb{R}\mathbf{\Psi}_1(H^1, H^1) + \mathbf{\Psi}_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2)$ (see Lemma 25) and because $\mathfrak{N}(2)$ is the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{N}(3)$.

By the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[H^1, X], H^1], Y\big) = \big\langle \mathbf{\Psi}_1(H^1, H^1), \mathbf{\Psi}_1(X, Y) \big\rangle - \big\langle \mathbf{\Psi}_1(H^1, Y), \mathbf{\Psi}_1(X, H^1) \big\rangle.$$

As observed in the proof of Proposition 32, we have $[[H^1, X], H^1] = X$. Since $\big(X, Y\big) = 0$, the left side of the above equality vanishes. Moreover, in view of (4.9) we have

$\langle \boldsymbol{\Psi}_1(H^1, Y), \boldsymbol{\Psi}_1(X, H^1) \rangle = 0$. Consequently, we have $\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, H^1) \rangle = 0$. Let $Z$ be an arbitrary element of $\mathfrak{B}_1$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[X, H^1], Y], Z\big) = \big\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, Z) \big\rangle - \big\langle \boldsymbol{\Psi}_1(X, Z), \boldsymbol{\Psi}_1(H^1, Y) \big\rangle.$$

Here we can easily verify that $[[X, H^1], Y] \in \mathfrak{sp}(2)$ and hence the left side of the above equality vanishes. By Proposition 30 (1), (2) we have $\boldsymbol{\Psi}_1(X, Z) = 0$ if $X \neq Z$. Hence $\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, Z) \rangle = 0$. On the other hand, if $X = Z$, then we have $\boldsymbol{\Psi}_1(X, Z) = \boldsymbol{\Psi}_1(X, X) = \boldsymbol{\Psi}_1(H^1, H^1) + \boldsymbol{\Psi}_1(H^1_1, H^1_1)$ (see Proposition 30). Hence by Proposition 28 (4) and the fact $\boldsymbol{\Psi}_1(H^1_1, H^1_1) \in \mathfrak{N}(2)$ we have $\langle \boldsymbol{\Psi}_1(X, Z), \boldsymbol{\Psi}_1(H^1, Y) \rangle = 0$. Therefore, in this case, we also obtain $\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, Z) \rangle = 0$. Since $Z$ is an arbitrary element of $\mathfrak{B}_1$, we have $\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, \mathfrak{m}_1) \rangle = 0$. In a similar way we can prove $\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(H^1, \mathfrak{m}_2) \rangle = 0$, showing (4.10). Accordingly, we get $\boldsymbol{\Psi}_1(X, Y) \in \mathfrak{N}(2)$ and hence $\boldsymbol{\Psi}_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$. □

Now let $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$. Take arbitrary elements $Z_1$, $Z_2 \in \mathfrak{sp}(2)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[X, Z_1], Y], Z_2\big) = \big\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_1(Z_1, Z_2) \big\rangle - \big\langle \boldsymbol{\Psi}_1(X, Z_2), \boldsymbol{\Psi}_1(Z_1, Y) \big\rangle.$$

By the results of Case $(a)$ and Case $(c)$ we have $\boldsymbol{\Psi}_1(Z_1, Z_2) = \boldsymbol{\Psi}_0(Z_1, Z_2)$, $\boldsymbol{\Psi}_1(X, Z_2) = \boldsymbol{\Psi}_0(X, Z_2)$ and $\boldsymbol{\Psi}_1(Y, Z_1) = \boldsymbol{\Psi}_0(Y, Z_1)$. Therefore we have

$$\big\langle \boldsymbol{\Psi}_1(X, Y), \boldsymbol{\Psi}_0(Z_1, Z_2) \big\rangle = \frac{1}{4}\big([[X, Z_1], Y], Z_2\big) + \big\langle \boldsymbol{\Psi}_0(X, Z_2), \boldsymbol{\Psi}_0(Z_1, Y) \big\rangle.$$

Since $\boldsymbol{\Psi}_0$ is a solution of the Gauss equation (2.3), we have

$$\big\langle \boldsymbol{\Psi}_0(X, Y), \boldsymbol{\Psi}_0(Z_1, Z_2) \big\rangle = \frac{1}{4}\big([[X, Z_1], Y], Z_2\big) + \big\langle \boldsymbol{\Psi}_0(X, Z_2), \boldsymbol{\Psi}_0(Z_1, Y) \big\rangle.$$

Hence, by subtraction, we have $\langle \boldsymbol{\Psi}_1(X, Y) - \boldsymbol{\Psi}_0(X, Y), \boldsymbol{\Psi}_0(Z_1, Z_2) \rangle = 0$. Here we note that $\boldsymbol{\Psi}_1(X, Y) - \boldsymbol{\Psi}_0(X, Y) \in \mathfrak{N}(2)$. Indeed, we have $\boldsymbol{\Psi}_1(X, Y) \in \mathfrak{N}(2)$ (see Lemma 35) and have $\boldsymbol{\Psi}_0(X, Y) \in \mathfrak{N}(2)$ by a simple calculation. Since $\boldsymbol{\Psi}_0(\mathfrak{sp}(2), \mathfrak{sp}(2)) = \mathfrak{N}(2)$, the above equality implies that $\boldsymbol{\Psi}_1(X, Y) - \boldsymbol{\Psi}_0(X, Y) = 0$, i.e., $\boldsymbol{\Psi}_1(X, Y) = \boldsymbol{\Psi}_0(X, Y)$. This completes the proof of $(e)$ in the case where $n = 3$.

Thus by the above case studies $(a) \sim (e)$ we get $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_0$, i.e., $\rho\boldsymbol{\Psi} = \boldsymbol{\Psi}_0$. This completes the proof of Theorem 10. □

**Remark 36.** As seen in the above discussion, we have proved Theorem 10 by utilizing the equality $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \boldsymbol{K}_0(X)$ for regular elements $X$ or for elements $X \in \mathcal{S}$. After we have established Theorem 10, we easily conclude that $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \boldsymbol{K}_0(X)$ holds for any element $X \in \mathfrak{sp}(n)$.

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Faculty of Integrated Arts and Sciences,
Hiroshima University
1-7-1 Kagamiyama, Higashi-Hiroshima City, Hiroshima, 739-8521
Japan
*E-mail address*: agaoka@mis.hiroshima-u.ac.jp

Faculty of Foreign Studies,
Osaka University of Foreign Studies
8-1-1 Aomadani-Higashi, Minoo City, Osaka, 562-8558
Japan
*E-mail address*: kaneda@osaka-gaidai.ac.jp