RIGIDITY OF THE CANONICAL ISOMETRIC IMBEDDING OF THE HERMITIAN SYMMETRIC SPACE $Sp(n)/U(n)$

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Abstract. In this paper we discuss the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$ of the Hermitian symmetric space $Sp(n)/U(n)$ into the Lie algebra $\mathfrak{sp}(n)$. We will show that if $n \geq 2$, then $\boldsymbol{f}_0$ is strongly rigid, i.e., for any isometric immersion $\boldsymbol{f}_1$ of a connected open set U of $Sp(n)/U(n)$ into $\mathfrak{sp}(n)$ there is a euclidean transformation a of $\mathfrak{sp}(n)$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

1. Introduction

In a series of our work [4], [5] and [7] we showed the strong rigidity of the canonical isometric imbeddings of the projective planes $P^2(\mathbb{CAY})$, $P^2(\mathbb{H})$ and the symplectic group $Sp(n)$. In this paper we will investigate the canonical isometric imbedding $\boldsymbol{f}_0$ of the Hermitian symmetric space $Sp(n)/U(n)$ $(n \geq 2)$ and establish the strong rigidity theorem for $\boldsymbol{f}_0$.

As is known, any Hermitian symmetric space M of compact type is isometrically imbedded into the Lie algebra $\mathfrak{g}$ of the holomorphic isometry group of M (see Lichnérowicz [15]). Thus, $Sp(n)/U(n)$ can be isometrically imbedded into $\mathfrak{sp}(n)$, which is the Lie algebra of the symplectic group $Sp(n)$. Identifying $\mathfrak{sp}(n)$ with the euclidean space $\mathbb{R}^{2n^2+n}$, we obtain an isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)/U(n)$ into $\mathbb{R}^{2n^2+n}$, which is called the *canonical isometric imbedding* of $Sp(n)/U(n)$. In [2] we proved that any open set of $Sp(n)/U(n)$ cannot be isometrically immersed into the euclidean space $\mathbb{R}^N$ with $N \leq \dim \mathfrak{sp}(n) - 1$. Accordingly, the canonical isometric imbedding $\boldsymbol{f}_0$ gives the least dimensional (local) isometric imbedding of $Sp(n)/U(n)$ into the euclidean space (see Corollary 2.5 of [2]).

In this paper we will prove

Theorem 1. *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $Sp(n)/U(n)$ $(n \geq 2)$ into the euclidean space $\mathfrak{sp}(n)$ $(= \mathbb{R}^{2n^2+n})$. Then $\boldsymbol{f}_0$ is strongly rigid, i.e., for any isometric immersion $\boldsymbol{f}_1$ of a connected open set U of $Sp(n)/U(n)$ into $\mathfrak{sp}(n)$ there is a euclidean transformation a of $\mathfrak{sp}(n)$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

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As for the rigidity on the canonical isometric imbeddings of connected irreducible Hermitian symmetric spaces M of compact type, the following results are known:

(1) $\boldsymbol{f}_0$ is globally rigid in the following sense: Let $\boldsymbol{f}_1$ be an isometric imbedding of M into $\mathfrak{g}$. If $\boldsymbol{f}_1$ is sufficiently close to $\boldsymbol{f}_0$ with respect to C^3-topology, then there is a euclidean transformation a of $\mathfrak{g}$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ (see Tanaka [17]).

(2) If M is not isomorphic to any complex projective space $P^n(\mathbb{C})$, then $\boldsymbol{f}_0$ is locally rigid in the following sense: Let U be a connected open set of M and let $\boldsymbol{f}_1$ be an isometric imbedding of U into $\mathfrak{g}$. If $\boldsymbol{f}_1$ is sufficiently close to $\boldsymbol{f}_0$ with respect to C^2-topology on U, then there is a euclidean transformation a of $\mathfrak{g}$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ holds on U (see Kaneda-Tanaka [12]).

We note that the topological condition on the mappings are removed in the statement of Theorem 1. In this sense Theorem 1 strengthens the rigidity theorem in [17] and [12] for the Hermitian symmetric space $Sp(n)/U(n)\,(n \geq 2)$.

The method of our proof is quite similar to the methods adopted in [4], [5] and [7]. We will solve the Gauss equation on $Sp(n)/U(n)$ in codimension $n^2 (= \dim \mathfrak{sp}(n) - \dim Sp(n)/U(n))$ and prove that any solution $\boldsymbol{\Psi}$ of the Gauss equation is Hermitian, i.e., $\boldsymbol{\Psi}(IX, IY) = \boldsymbol{\Psi}(X, Y)$. This fact, together with the criterion on the isometric imbeddings of almost Hermitian manifolds (Theorem 5), indicates that any solution of the Gauss equation is equivalent to the second fundamental form of $\boldsymbol{f}_0$. Therefore by the congruence theorem obtained in [4] (see Theorem 3 below) we can establish Theorem 1.

Throughout this paper we will assume the differentiability of class C^∞. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [11]. For the quaternion numbers and the symplectic group $Sp(n)$, see Chevalley [9].

2. The Gauss equation and rigidity of isometric imbeddings

Let M be a Riemannian manifold and let $T(M)$ be the tangent bundle of M. We denote by ν the Riemannian metric of M and by R the Riemannian curvature tensor of type $(1,3)$ with respect to ν. We also denote by C the Riemannian curvature tensor of type $(0,4)$, which is, at each point $p \in M$, given by

$$C(x,y,z,w) = -\nu\big(R(x,y)z, w\big), \qquad x,y,z,w \in T_p(M).$$

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\mathbb{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $S^2T_p^*(M) \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $T_p(M)$. We call the following equation on $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$ the *Gauss equation* at $p \in M$ modeled on $\boldsymbol{N}$:

$$C(x,y,z,w) = \big\langle \boldsymbol{\Psi}(x,z), \boldsymbol{\Psi}(y,w)\big\rangle - \big\langle \boldsymbol{\Psi}(x,w), \boldsymbol{\Psi}(y,z)\big\rangle, \tag{2.1}$$

where x, y, z, $w \in T_p(M)$. We denote by $\mathcal{G}_p(M, \boldsymbol{N})$ the set of all solutions of (2.1), which is called the *Gaussian variety* at $p \in M$ modeled on $\boldsymbol{N}$. Let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2T_p^*(M) \otimes \boldsymbol{N}$ by

$$(h\boldsymbol{\Psi})(x, y) = h(\boldsymbol{\Psi}(x, y)),$$

where $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$, x, $y \in T_p(M)$. We say that two elements $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are *equivalent* if there is an element $h \in O(\boldsymbol{N})$ such that $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$. It is easily seen that if $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are equivalent and $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$, then $\boldsymbol{\Psi}' \in \mathcal{G}_p(M, \boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}_p(M, \boldsymbol{N})$ is *EOS* if $\mathcal{G}_p(M, \boldsymbol{N}) \neq \emptyset$ and if it consists of essentially one solution, i.e., any solutions of the Gauss equation (2.1) are equivalent to each other under the action of $O(\boldsymbol{N})$. We proved

Proposition 2 ([4], p. 334)**.** *Let M be a Riemannian manifold and $p \in M$. Let $\boldsymbol{N}$ be a euclidean vector space such that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. Then*:

(1) *Let $\boldsymbol{\Psi}$ be an arbitrary element of $\mathcal{G}_p(M, \boldsymbol{N})$. Then, the vectors $\boldsymbol{\Psi}(x, y)\,(x, y \in T_p(M))$ span the whole space $\boldsymbol{N}$.*

(2) *Let $\boldsymbol{N}_1$ be a euclidean vector space. Then*:
  - (2*a*) $\mathcal{G}_p(M, \boldsymbol{N}_1) = \emptyset$ *if* $\dim \boldsymbol{N}_1 < \dim \boldsymbol{N}$;
  - (2*b*) $\mathcal{G}_p(M, \boldsymbol{N}_1)$ *is EOS if* $\dim \boldsymbol{N}_1 = \dim \boldsymbol{N}$;
  - (2*c*) $\mathcal{G}_p(M, \boldsymbol{N}_1)$ *is not EOS if* $\dim \boldsymbol{N}_1 > \dim \boldsymbol{N}$.

We say that a Riemannian manifold M is *formally rigid* in codimension r if there is a euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = r$ such that the Gaussian variety $\mathcal{G}_p(M, \boldsymbol{N})$ modeled on $\boldsymbol{N}$ is EOS at each $p \in M$. In [4] we have obtained the following rigidity theorem for formally rigid Riemannian manifolds:

Theorem 3 ([4], pp. 335–336)**.** *Let M be an m-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the euclidean space $\mathbb{R}^N$. Assume*:

(1) *M is connected*;

(2) *M is formally rigid in codimension $r = N - m$.*

Then, any isometric immersion $\boldsymbol{f}_1$ of M into the euclidean space $\mathbb{R}^N$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\mathbb{R}^N$, i.e., there exists a euclidean transformation a of $\mathbb{R}^N$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$.

In the subsequent sections we will prove

Theorem 4. *The Hermitian symmetric space $Sp(n)/U(n)\,(n \geq 2)$ is formally rigid in codimension $n^2\,(= \dim \mathfrak{u}(n))$.*

If Theorem 4 is true, then it is easily seen that Theorem 1 immediately follows from Theorem 3.

Remark 1. We note that, in the case $n = 1$, Theorem 4 is not true. In this case we have $Sp(1)/U(1) \cong S^2$ and the canonical isometric imbedding $\boldsymbol{f}_0$ coincides with the standard isometric imbedding of S^2 into $\mathbb{R}^3$. Consequently, $\boldsymbol{f}_0$ is globally rigid (remember the rigidity theorem for ovaloids by Cohn-Vossen[10]). However, it is not locally rigid, i.e., there are infinitely many non-equivalent surfaces of revolution possessing constant positive curvature. Therefore, the Gauss equation in codimension 1 admits infinitely many non-equivalent solutions corresponding to the second fundamental forms of these surfaces. For details, see Spivak [16].

3. The Gauss equation on almost Hermitian manifolds

For the proof of Theorem 4 we start from a general setting. Let M be an even dimensional Riemannian manifold with Riemannian metric ν. Assume that there is an almost complex structure I on M such that $\nu\big(Ix, Iy\big) = \nu\big(x, y\big)$ $(x, y \in T_p(M))$ at each $p \in M$. Then M is called an *almost Hermitian manifold*.

Let M be an almost Hermitian manifold and $p \in M$. Let $\boldsymbol{N}$ be a euclidean vector space. An element $\boldsymbol{\Psi} \in S^2T_p^* \otimes \boldsymbol{N}$ is called *Hermitian* if $\boldsymbol{\Psi}(IX, IY) = \boldsymbol{\Psi}(X, Y)$ holds for any X, $Y \in T_p(M)$. In what follows we will consider the case where the Gauss equation (2.1) admits a Hermitian solution. We will prove

Theorem 5. *Let M be an almost Hermitian manifold and $\boldsymbol{N}$ a euclidean vector space. Let $\mathcal{G}_p(M, \boldsymbol{N})$ be the Gaussian variety at $p \in M$ modeled on $\boldsymbol{N}$. Assume*:

(1) $\mathcal{G}_p(M, \boldsymbol{N}) \neq \emptyset$;

(2) *Any solution $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$ is Hermitian.*

Then, $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS.

Let M be a $2m$-dimensional almost Hermitian manifold and let $p \in M$. For simplicity, we set $\boldsymbol{T} = T_p(M)$. Let $\boldsymbol{T}^{\mathbb{C}} = \boldsymbol{T} + \sqrt{-1}\boldsymbol{T}$ be the complexification of $\boldsymbol{T}$. By $\overline{X}$ we denote the complex conjugate of $X \in \boldsymbol{T}^{\mathbb{C}}$ with respect to $\boldsymbol{T}$. The almost complex structure I is extended to a $\mathbb{C}$-linear endomorphism of $\boldsymbol{T}^{\mathbb{C}}$, which is also denoted by I. Set

$$\boldsymbol{T}^{1,0} = \{Z \in \boldsymbol{T}^{\mathbb{C}} \mid IZ = \sqrt{-1}Z\}, \qquad \boldsymbol{T}^{0,1} = \{Z \in \boldsymbol{T}^{\mathbb{C}} \mid IZ = -\sqrt{-1}Z\}.$$

Then, as is known, $\boldsymbol{T}^{\mathbb{C}} = \boldsymbol{T}^{1,0} + \boldsymbol{T}^{0,1}$ (direct sum) and $\boldsymbol{T}^{0,1} = \overline{\boldsymbol{T}^{1,0}}$; $\boldsymbol{T}^{1,0} = \overline{\boldsymbol{T}^{0,1}}$. Take a basis $\{Z_1, \cdots, Z_m\}$ of $\boldsymbol{T}^{1,0}$ and put $Z_{\bar{i}} = \overline{Z_i}\,(1 \leq i \leq m)$. Then the set $\{Z_i, Z_{\bar{i}}\,(1 \leq i \leq m)\}$ forms a basis of $\boldsymbol{T}^{\mathbb{C}}$. In the following we will fix such a basis $\{Z_i, Z_{\bar{i}}\,(1 \leq i \leq m)\}$ and rewrite the Gauss equation (2.1).

As usual, the Riemannian curvature is extended to a tensor of type $(0,4)$ on $\boldsymbol{T}^{\mathbb{C}}$. Define $C_{abcd} \in \mathbb{C}$ by setting

$$C_{abcd} = C(Z_a, Z_b, Z_c, Z_d),$$

where the suffices a, b, c, d run through the range $1, \ldots, m, \bar{1}, \ldots, \bar{m}$. We also extend an element $\boldsymbol{\Psi} \in S^2\boldsymbol{T}^* \otimes \boldsymbol{N}$ to an element of $S^2\boldsymbol{T}^{\mathbb{C}*} \otimes \boldsymbol{N}^{\mathbb{C}}$, where $\boldsymbol{N}^{\mathbb{C}} = \boldsymbol{N} + \sqrt{-1}\boldsymbol{N}$ denotes the complexification of a euclidean vector space $\boldsymbol{N}$. Define vectors $\boldsymbol{\Psi}_{ab} \in \boldsymbol{N}^{\mathbb{C}}$ by setting

$$\boldsymbol{\Psi}_{ab} = \boldsymbol{\Psi}(Z_a, Z_b), \qquad a, b = 1, \ldots, m, \bar{1}, \ldots, \bar{m}.$$

Then we easily have

$$\overline{\boldsymbol{\Psi}_{ab}} = \boldsymbol{\Psi}_{\bar{a}\bar{b}}, \qquad a, b = 1, \ldots, m, \bar{1}, \ldots, \bar{m},$$

where for an element $\boldsymbol{v} \in \boldsymbol{N}^{\mathbb{C}}$ we mean by $\bar{\boldsymbol{v}}$ the complex conjugate of $\boldsymbol{v}$ with respect to $\boldsymbol{N}$ and we promise $\bar{\bar{i}} = i$ for $i = 1, \ldots, m$.

By use of C_{abcd} and $\boldsymbol{\Psi}_{ab}$ we can rewrite the Gauss equation (2.1) as follows:

$$C_{abcd} = \big\langle \boldsymbol{\Psi}_{ac}, \boldsymbol{\Psi}_{bd} \big\rangle - \big\langle \boldsymbol{\Psi}_{ad}, \boldsymbol{\Psi}_{bc} \big\rangle, \qquad a, b, c, d = 1, \ldots, m, \bar{1}, \ldots, \bar{m}, \tag{3.1}$$

where $\big\langle\ ,\ \big\rangle$ means the symmetric bilinear form on $\boldsymbol{N}^{\mathbb{C}}$ which is a natural extension of the inner product of $\boldsymbol{N}$. We now prove

Lemma 6. *Let $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$. Assume that $\boldsymbol{\Psi}$ is Hermitian. Then*

$$C_{i\bar{k}\bar{j}l} = \big\langle \boldsymbol{\Psi}_{i\bar{j}}, \boldsymbol{\Psi}_{\bar{k}l} \big\rangle, \qquad 1 \le i, j, k, l \le m. \tag{3.2}$$

Proof. Let i, j, k and l be integers such that $1 \le i, j, k, l \le m$. Putting $a = i$, $b = \bar{k}$, $c = \bar{j}$ and $d = l$ into (3.1), we have

$$C_{i\bar{k}\bar{j}l} = \big\langle \boldsymbol{\Psi}_{i\bar{j}}, \boldsymbol{\Psi}_{\bar{k}l} \big\rangle - \big\langle \boldsymbol{\Psi}_{il}, \boldsymbol{\Psi}_{\bar{k}\bar{j}} \big\rangle.$$

Since $\boldsymbol{\Psi}$ is Hermitian, we have $\boldsymbol{\Psi}_{il} = \boldsymbol{\Psi}_{\bar{k}\bar{j}} = 0$. Hence we get (3.2). □

Let us define a Hermitian inner product $\big(\ ,\ \big)$ of $\boldsymbol{N}^{\mathbb{C}}$ by setting

$$\big(Y, Y'\big) = \big\langle Y, \overline{Y'} \big\rangle, \qquad Y, Y' \in \boldsymbol{N}^{\mathbb{C}}.$$

Then $\boldsymbol{N}^{\mathbb{C}}$ is considered as a Hermitian vector space.

We now define a quadratic form $\widehat{\mathcal{C}(p)}$ on $\boldsymbol{T}^{1,0} \otimes \overline{\boldsymbol{T}^{1,0}}$ by

$$\widehat{\mathcal{C}(p)}(X \otimes \overline{Y}, Z \otimes \overline{W}) = C(X, \overline{Z}, \overline{Y}, W), \qquad X \otimes \overline{Y},\ Z \otimes \overline{W} \in \boldsymbol{T}^{1,0} \otimes \overline{\boldsymbol{T}^{1,0}}.$$

By $\mathcal{C}(p)$ we denote the matrix corresponding to $\widehat{\mathcal{C}(p)}$ with respect to the basis $\{Z_i \otimes Z_{\bar{j}}\ (1 \le i, j \le m)\}$ of $\boldsymbol{T}^{1,0} \otimes \overline{\boldsymbol{T}^{1,0}}$. As is easily seen, $\mathcal{C}(p) = (\mathcal{C}(p)_{\alpha\beta})$ is a complex square matrix of degree m^2, where Greek letters α, $\beta, \ldots$ run over the pairs of indices $\{i\bar{j}\}$ $(i, j = 1, \ldots, m)$ and

$$\mathcal{C}(p)_{\alpha\beta} = C_{i\bar{k}\bar{j}l}, \qquad \alpha = \{i\bar{j}\},\ \beta = \{k\bar{l}\}.$$

It is easily checked that $\mathcal{C}(p)$ is a Hermitian matrix, i.e., ${}^t\mathcal{C}(p) = \overline{\mathcal{C}(p)}$. Moreover, the rank of $\mathcal{C}(p)$ and the cardinal number of positive or negative eignevalues of $\mathcal{C}(p)$ do not depend on the choice of the basis $\{Z_i\}$ of $\boldsymbol{T}^{1,0}$.

Now, let $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$. Assume that $\boldsymbol{\Psi}$ is Hermitian. Then by (3.2) we have

$$\mathcal{C}(p)_{\alpha\beta} = \big(\boldsymbol{\Psi}_\alpha, \boldsymbol{\Psi}_\beta\big), \qquad \alpha, \beta = \{1\bar{1}\}, \dots, \{i\bar{j}\}, \dots, \{m\bar{m}\}, \tag{3.3}$$

where we write $\boldsymbol{\Psi}_\alpha = \boldsymbol{\Psi}_{i\bar{j}}$ when $\alpha = \{i\bar{j}\}$. The equality (3.3) indicates that $\mathcal{C}(p)$ is nothing but the Gram matrix of the vectors $\{\boldsymbol{\Psi}_\alpha\}_\alpha$ with respect to $\big(\, , \, \big)$. Therefore, $\mathcal{C}(p)$ must be positive semi-definite and $\operatorname{rank}(\mathcal{C}(p)) = \dim_{\mathbb{C}}(\sum_\alpha \mathbb{C}\boldsymbol{\Psi}_\alpha)$, where α runs through the indices $\{1\bar{1}\}, \dots, \{m\bar{m}\}$. Let $\boldsymbol{N}_{\boldsymbol{\Psi}}$ be the subspace of $\boldsymbol{N}$ spanned by the vectors $\boldsymbol{\Psi}(X, Y)$, where $X, Y \in \boldsymbol{T}$. Then we easily have $\boldsymbol{N}_{\boldsymbol{\Psi}}^{\mathbb{C}} = \sum_\alpha \mathbb{C}\boldsymbol{\Psi}_\alpha$. Hence $\dim \boldsymbol{N}_{\boldsymbol{\Psi}} = \dim_{\mathbb{C}}(\sum_\alpha \mathbb{C}\boldsymbol{\Psi}_\alpha)$. Therefore, we get

Lemma 7. *Let $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$. Assume that $\boldsymbol{\Psi}$ is Hermitian. Then $\mathcal{C}(p)$ is positive semi-definite and*

$$\dim \boldsymbol{N}_{\boldsymbol{\Psi}} = \operatorname{rank}(\mathcal{C}(p)).$$

Consequently, $\mathcal{G}_p(M, \boldsymbol{N})$ does not contain any Hermitian element if one of the following conditions are satisfied:

(1) *$\mathcal{C}(p)$ has at least one negative eigenvalue;*
(2) *$\dim \boldsymbol{N} < \operatorname{rank}(\mathcal{C}(p))$.*

Example 1. Let M be a Kähler manifold of constant holomorphic sectional curvature $c\,(\neq 0)$ and $p \in M$. Let $(z_1, \dots, z_m)$ be a complex local coordinate system of M around p. Put $Z_i = \partial/\partial z_i \, (1 \leq i \leq m)$. Then we get a basis $\{Z_i\}_{1 \leq i \leq m}$ of $\boldsymbol{T}^{1,0}$. By use of the basis $\{Z_i, Z_{\bar{i}} \, (1 \leq i \leq m)\}$ the curvature tensor C of M can be written as

$$C_{i\bar{k}\bar{j}l} = \frac{1}{2} c \,(\nu_{i\bar{k}}\nu_{\bar{j}l} + \nu_{i\bar{j}}\nu_{\bar{k}l}), \qquad 1 \leq i, j, k, l \leq m,$$

where we set $\nu_{i\bar{k}} = \nu(Z_i, Z_{\bar{k}}) \, (1 \leq i, k \leq m)$ (see Kobayashi-Nomizu [14]). By a suitable change of the coordinate $(z_1, \dots, z_m)$ we may assume that $\nu_{i\bar{k}} = \delta_{ik} \, (1 \leq i, k \leq m)$ at p, where δ means the Kronecker delta. Consequently, the component $\mathcal{C}(p)_{\alpha\beta}$ of the matrix $\mathcal{C}(p)$ is given by

$$\mathcal{C}(p)_{\alpha\beta} = \frac{1}{2} c \,(\delta_{\alpha\beta} + \delta_{\alpha\bar{\alpha}}\delta_{\beta\bar{\beta}}), \qquad \alpha, \beta = \{1\bar{1}\}, \dots, \{i\bar{j}\}, \dots, \{m\bar{m}\},$$

where $\{i\bar{j}\} = \{k\bar{l}\}$ means $i = k$ and $j = l$; $\overline{\{i\bar{j}\}} = \{j\bar{i}\}$, $\overline{\{k\bar{l}\}} = \{l\bar{k}\}$. Therefore, we have $\operatorname{rank}(\mathcal{C}(p)) = m^2$, because $c \neq 0$. Further, if $c < 0$ (resp. $c > 0$), then $\mathcal{C}(p)$ is negative (resp. positive) definite. Accordingly, $\mathcal{G}_p(M, \boldsymbol{N})$ does not contain any Hermitian element in case $c < 0$ or $\dim \boldsymbol{N} < m^2$.

With these preparations we prove Theorem 5.

Proof of Theorem 5. First suppose that $\dim \boldsymbol{N} > \operatorname{rank}(\mathcal{C}(p))$. Let $\boldsymbol{\Psi}$ be an arbitrary element of $\mathcal{G}_p(M, \boldsymbol{N})$. Then by Lemma 7 we have $\dim \boldsymbol{N}_{\boldsymbol{\Psi}} < \dim \boldsymbol{N}$. Take a non-zero vector $\boldsymbol{n} \in \boldsymbol{N}$ such that $\langle \boldsymbol{N}_{\boldsymbol{\Psi}}, \boldsymbol{n} \rangle = 0$ and take a non-zero covector $\xi \in \boldsymbol{T}^*$. Set $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi} + \xi^2 \otimes \boldsymbol{n}$. Then it is easily verified that $\boldsymbol{\Psi}_1 \in \mathcal{G}_p(M, \boldsymbol{N})$ and that $\boldsymbol{\Psi}_1$ is not Hermitian. This contradicts the assumption (2). Consequently, we have $\dim \boldsymbol{N} = \operatorname{rank}(\mathcal{C}(p))$. Moreover, we have $\boldsymbol{N}_{\boldsymbol{\Psi}} = \boldsymbol{N}$ for any $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$.

We now prove that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. Let $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in \mathcal{G}_p(M, \boldsymbol{N})$. Since $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ are Hermitian, they satisfy the equality (3.3). Hence we have

$$\big(\boldsymbol{\Psi}_\alpha, \boldsymbol{\Psi}_\beta\big) = \big(\boldsymbol{\Psi}'_\alpha, \boldsymbol{\Psi}'_\beta\big), \qquad \alpha, \beta = \{1\bar{1}\}, \dots, \{i\bar{j}\}, \dots, \{m\bar{m}\}.$$

Since $\boldsymbol{N}_{\boldsymbol{\Psi}} = \boldsymbol{N}$, the vectors $\{\boldsymbol{\Psi}_\alpha\}_\alpha$ span the whole $\boldsymbol{N}^{\mathbb{C}}$. By an elementary linear algebra we know that there is a unitary transformation h of $\boldsymbol{N}^{\mathbb{C}}$ satisfying $\boldsymbol{\Psi}'_\alpha = h(\boldsymbol{\Psi}_\alpha)\,(\alpha = \{1\bar{1}\}, \dots, \{i\bar{j}\}, \dots, \{m, \bar{m}\})$. Let $\alpha = \{i\bar{j}\}$. Then we have

$$h(\overline{\boldsymbol{\Psi}_\alpha}) = h(\boldsymbol{\Psi}_{j\bar{i}}) = \boldsymbol{\Psi}'_{j\bar{i}} = \overline{\boldsymbol{\Psi}'_\alpha} = \overline{h(\boldsymbol{\Psi}_\alpha)}.$$

Consequently, we have $h(\boldsymbol{N}) \subset \boldsymbol{N}$. Hence h is an orthogonal transformation of $\boldsymbol{N}$ and satisfies $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$. This shows that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. □

Remark 2. Let $\boldsymbol{N}$ be a euclidean vector space. Assume that $\mathcal{G}_p(M, \boldsymbol{N})$ satisfies the conditions (1) and (2) in Theorem 5. Let $\boldsymbol{N}'$ be another euclidean vector space such that $\dim \boldsymbol{N}' = \dim \boldsymbol{N}$. Then, $\mathcal{G}_p(M, \boldsymbol{N}')$ also satisfies the conditions (1) and (2) in Theorem 5. To observe this, take an isometric isomorphism $\varphi\colon \boldsymbol{N} \longrightarrow \boldsymbol{N}'$ and define a linear mapping $S^2T_p^* \otimes \boldsymbol{N} \ni \boldsymbol{\Psi} \longmapsto \widehat{\boldsymbol{\Psi}} \in S^2T_p^* \otimes \boldsymbol{N}'$ by $\widehat{\boldsymbol{\Psi}}(X, Y) = \varphi(\boldsymbol{\Psi}(X, Y))\,(X, Y \in T_p(M))$. Then the following assertions can be easily verified:

(1) $\widehat{\boldsymbol{\Psi}}$ is Hermitian if and only if $\boldsymbol{\Psi}$ is Hermitian;

(2) $\widehat{\boldsymbol{\Psi}} \in \mathcal{G}_p(M, \boldsymbol{N}')$ if and only if $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$.

Thus, we note that the conditions (1) and (2) in Theorem 5 are the conditions only related to the dimension of the eulidean space $\boldsymbol{N}$. As seen in the proof of Theorem 5, $\dim \boldsymbol{N}$ equals $\operatorname{rank}(\mathcal{C}(p))$, which is uniquely determined by the curvature of M at p.

4. The canonical isometric imbeddings of Hermitian symmetric spaces of compact type

We now review the canonical isometric imbeddings of Hermitian symmetric spaces of compact type defined by Lichnérowicz [15]. Let M be an almost Hermitian manifold. A mapping g of M into itself is called *holomorphic* if g satisfies $g_* \circ I = I \circ g_*$. A connected almost Hermitian manifold M is called a *Hermitian symmetric space* if each

$p \in M$ is an isolated fixed point of an involutive holomorphic isometry of M. Utilizing the identity component G of the holomorphic isometry group of M, we can represent M as a Riemannian symmetric space G/K, where K is an isotropy group at a suitable point $o \in M$; usually o is called the *origin* of G/K (see Helgason [11]). We say a Hermitian symmetric space G/K is *of compact type* if the Lie algebra $\mathfrak{g}$ of G is compact and semisimple.

Let $M = G/K$ be a Hermitian symmetric space of compact type. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ with respect to the Riemannian symmetric pair (G, K). As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(M)$. It is known that there is an element $I_0 \in \mathfrak{k}$ such that: (i) I_0 belongs to the center of $\mathfrak{k}$; (ii) $\mathrm{ad}\,(I_0)|_{\mathfrak{m}}$ coincides with the almost complex structure I at o (see [14], [11]). Consider the $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$ passing through I_0, i.e., $\mathrm{Ad}(G)I_0 \subset \mathfrak{g}$. Since $\mathrm{Ad}(K)I_0 = I_0$, we get a differential mapping

$$\boldsymbol{f}_0 \colon G/K \ni gK \longmapsto \mathrm{Ad}(g)I_0 \in \mathfrak{g}.$$

We may regard $\mathfrak{g}$ as a euclidean vector space with a suitable $\mathrm{Ad}(G)$-invariant inner product. The induced Riemannian metric ν of G/K via $\boldsymbol{f}_0$ is easily understood to be G-invariant. The mapping $\boldsymbol{f}_0$ is called the *canonical isometric imbedding* of $M = G/K$.

Let ∇ be the Riemannian connection on $M = G/K$ associated with ν. By $\nabla \boldsymbol{f}_0$ (resp. $\nabla\nabla \boldsymbol{f}_0$) we denote the first order (resp. second order) covariant derivative of the canonical isometric imbedding $\boldsymbol{f}_0$. The second order covariant derivative $\nabla\nabla \boldsymbol{f}_0$ is called the *second fundamental form* of the canonical isometric imbedding $\boldsymbol{f}_0$. In view of Tanaka [17], Kaneda-Tanaka [12] we know that at the origin o, $\nabla \boldsymbol{f}_0$ and $\nabla\nabla \boldsymbol{f}_0$ are given as follows:

$$\nabla_X \boldsymbol{f}_0 = [X, I_0] = -IX, \quad X \in \mathfrak{m}; \tag{4.1}$$

$$\nabla_X \nabla_Y \boldsymbol{f}_0 = [X, [Y, I_0]] = -[X, IY], \quad X, Y \in \mathfrak{m}. \tag{4.2}$$

Let $\boldsymbol{T}$ (resp. $\boldsymbol{N}$) be the tangent (resp. normal) vector space of $\boldsymbol{f}_0(G/K)$ at I_0 ($= \boldsymbol{f}_0(o) \in \mathfrak{g}$). By (4.1) we know that the tangent space $\boldsymbol{T}$, which is generated by the first order covariant derivatives of $\boldsymbol{f}_0$ at o, coincides with $\mathfrak{m}$. Consequently, the normal vector space $\boldsymbol{N}$ at o is given by $\boldsymbol{N} = \mathfrak{k}$. Similarly, by (4.2) we know that the value of the second order covariant derivative $\nabla_X \nabla_Y \boldsymbol{f}_0\,(X, Y \in \mathfrak{m})$ belongs to the normal vector space $\boldsymbol{N} = \mathfrak{k}$. As for the second fundamental form we have

Proposition 8 ([17], [12]). *Let G/K be a Hermitian symmetric space of compact type and let $\boldsymbol{f}_0$ be the canonical isometric imbedding of G/K into $\mathfrak{g}$. Then the second fundamental form $\boldsymbol{\Psi}_0 \in S^2\mathfrak{m}^* \otimes \mathfrak{k}$ of $\boldsymbol{f}_0$ at the origin o satisfies the following*

(1) $\boldsymbol{\Psi}_0 \in \mathcal{G}_o(G/K, \mathfrak{k})$;

(2) *The vectors $\boldsymbol{\Psi}_0(X, Y)\,(X, Y \in \mathfrak{m})$ span the whole $\mathfrak{k}$;*

(3) $\boldsymbol{\Psi}_0$ *is Hermitian, i.e., $\boldsymbol{\Psi}_0(IX, IY) = \boldsymbol{\Psi}_0(X, Y)$ for $X, Y \in \mathfrak{m}$.*

By this proposition we have

Proposition 9. *Let G/K be a Hermitian symmetric space of compact type. Then for each $p \in G/K$ the following assertions hold*:

(1) $\operatorname{rank}(\mathcal{C}(p)) = \dim \mathfrak{k}$;
(2) *Let $\boldsymbol{N}$ be a euclidean vector space with* $\dim \boldsymbol{N} = \dim \mathfrak{k}$. *Then, $\mathcal{G}_p(G/K, \boldsymbol{N})$ is EOS if and only if any element $\boldsymbol{\Psi} \in \mathcal{G}_o(G/K, \mathfrak{k})$ is Hermitian.*

Proof. By Proposition 8 and Lemma 7 we immediately know that $\operatorname{rank}(\mathcal{C}(o)) = \dim \mathfrak{k}$. Then, by homogeneity of G/K, we have (1). Also, by homogeneity, we easily see that $\mathcal{G}_p(G/K, \boldsymbol{N})$ is EOS if and only if $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS. Note that $\mathcal{G}_o(G/K, \mathfrak{k})$ contains a Hermitian element $\boldsymbol{\Psi}_0$. Hence, if $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS, then any element $\boldsymbol{\Psi} \in \mathcal{G}_o(G/K, \mathfrak{k})$ is Hermitian. The converse part follows from Theorem 5. □

Remark 3. Let G/K be a Hermitian symmetric space of compact type and let $p \in G/K$. Then, the equality $\operatorname{rank}(\mathcal{C}(p)) = \dim \mathfrak{k}$ in Proposition 9 indicates that $\dim \mathfrak{k}$ is the least dimension of a euclidean vector space $\boldsymbol{N}$ such that $\mathcal{G}_p(G/K, \boldsymbol{N})$ contains a Hermitian element. In fact, if $\boldsymbol{N}_1$ is a euclidean vector space with $\dim \boldsymbol{N}_1 < \dim \mathfrak{k}$, then $\mathcal{G}_p(G/K, \boldsymbol{N}_1)$ does not contain any Hermitian element (see Lemma 7). However, we note that this fact does not imply $\mathcal{G}_p(G/K, \boldsymbol{N}_1) = \emptyset$. Agaoka [1] proved that for the complex projective space $P^n(\mathbb{C})\,(n \geq 2)$, $\mathcal{G}_o(P^n(\mathbb{C}), \boldsymbol{N}_1) \neq \emptyset$ when $\dim \boldsymbol{N}_1 = n^2 - 1$. We note that in this case we have $\dim \mathfrak{k} = \dim \mathfrak{u}(n) = n^2$ and hence $\mathcal{G}_o(P^n(\mathbb{C}), \mathfrak{k})$ is not EOS. It seems to the authors that this is a special case. For the other irreducible Hermitian symmetric space G/K except $P^n(\mathbb{C})\,(n \geq 1)$, such as the complex Grassmann manifold $G^{p,q}(\mathbb{C})\,(p \geq q \geq 2)$, the complex quadric $Q^n(\mathbb{C})\,(n \geq 4)$, etc., we conjecture that the Gaussian variety $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS. As will be seen in the following sections, our conjecture is true for the Hermitian symmetric space $Sp(n)/U(n)\,(n \geq 2)$. In Table 1 we show all *irreducible* Hermitian symmetric spaces G/K of compact type and related data:

Table 1. Irreducible Hermitian symmetric spaces of compact type

G/K	$\operatorname{rank}(G/K)$	$\dim G/K$	$\dim \mathfrak{g}$	$\dim \mathfrak{k}$
$P^n(\mathbb{C})\,(n \geq 1)$	1	$2n$	$n^2 + 2n$	n^2
$G^{p,q}(\mathbb{C})\,(p \geq q \geq 2)$	q	$2pq$	$(p+q)^2 - 1$	$p^2 + q^2 - 1$
$Q^n(\mathbb{C})\,(n \geq 5)$	2	$2n$	$\frac{1}{2}(n+1)(n+2)$	$\frac{1}{2}n(n-1) + 1$
$SO(2n)/U(n)\,(n \geq 5)$	$[n/2]$	$n^2 - n$	$2n^2 - n$	n^2
$Sp(n)/U(n)\,(n \geq 1)$	n	$n^2 + n$	$2n^2 + n$	n^2
$E_6/Spin(10) \cdot SO(2)$	2	32	78	46
$E_7/E_6 \cdot SO(2)$	3	54	133	79

5. The Hermitian symmetric space $Sp(n)/U(n)$

Let $\mathbb{H}$ be the field of quaternion numbers. As is well-known, $\mathbb{H}$ is an associative algebra over the field $\mathbb{R}$ of real numbers generated by $\mathbf{1}$ ($\in \mathbb{R}$) and three elements $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ satisfying the following multiplication rule:

(1) $\mathbf{1}\boldsymbol{i} = \boldsymbol{i}\mathbf{1} = \boldsymbol{i}, \quad \mathbf{1}\boldsymbol{j} = \boldsymbol{j}\mathbf{1} = \boldsymbol{j}, \quad \mathbf{1}\boldsymbol{k} = \boldsymbol{k}\mathbf{1} = \boldsymbol{k}$;
(2) $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathbf{1}$;
(3) $\boldsymbol{i}\boldsymbol{j} = -\boldsymbol{j}\boldsymbol{i} = \boldsymbol{k}, \quad \boldsymbol{j}\boldsymbol{k} = -\boldsymbol{k}\boldsymbol{j} = \boldsymbol{i}, \quad \boldsymbol{k}\boldsymbol{i} = -\boldsymbol{i}\boldsymbol{k} = \boldsymbol{j}$.

Let $\boldsymbol{q} \in \mathbb{H}$. Then $\boldsymbol{q}$ is written as $\boldsymbol{q} = q_0\mathbf{1} + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. We define the norm $|\boldsymbol{q}|$, the real part $\mathrm{Re}(\boldsymbol{q})$ and the conjugate $\overline{\boldsymbol{q}}$ by

$$|\boldsymbol{q}|^2 = \sum_{i=0}^{3} q_i^2; \qquad \mathrm{Re}(\boldsymbol{q}) = q_0; \qquad \overline{\boldsymbol{q}} = q_0\mathbf{1} - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}.$$

Then we easily have $|\overline{\boldsymbol{q}}| = |\boldsymbol{q}|$, $\mathrm{Re}(\boldsymbol{q}) = \mathrm{Re}(\overline{\boldsymbol{q}})$, $\overline{\overline{\boldsymbol{q}}} = \boldsymbol{q}$ and $\boldsymbol{q}\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}\boldsymbol{q} = |\boldsymbol{q}|^2$. Further, we have

$$|\boldsymbol{q}\boldsymbol{q}'| = |\boldsymbol{q}||\boldsymbol{q}'|; \qquad \mathrm{Re}(\boldsymbol{q}\boldsymbol{q}') = \mathrm{Re}(\boldsymbol{q}'\boldsymbol{q}); \qquad \overline{\boldsymbol{q}\boldsymbol{q}'} = \overline{\boldsymbol{q}'}\,\overline{\boldsymbol{q}}, \qquad \forall \boldsymbol{q}, \boldsymbol{q}' \in \mathbb{H}.$$

By the identification $a \in \mathbb{R}$ with $a\mathbf{1} \in \mathbb{H}$ the field $\mathbb{R}$ is canonically considered as a subfield of $\mathbb{H}$. By the identification $a + b\sqrt{-1} \longmapsto a + b\boldsymbol{i}$ $(a, b \in \mathbb{R})$ we may regard the field $\mathbb{C}$ of complex numbers as the subfield $\mathbb{R} + \mathbb{R}\boldsymbol{i}$ of $\mathbb{H}$. In this meaning we will write $\mathbb{C} = \mathbb{R} + \mathbb{R}\boldsymbol{i}$. The real part and the conjugate of a quaternion number defined above are compatible with the usual one defined on $\mathbb{C}$.

For later convenience we set $\mathbb{D} = \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$. Then we easily have

$$\mathbb{H} = \mathbb{C} + \mathbb{D} \text{ (direct sum)}; \quad \mathbb{C}\mathbb{C} = \mathbb{D}\mathbb{D} = \mathbb{C}, \quad \mathbb{C}\mathbb{D} = \mathbb{D}\mathbb{C} = \mathbb{D}.$$

We now define a bracket in $\mathbb{H}$ by $[\boldsymbol{q}, \boldsymbol{q}'] = \boldsymbol{q}\boldsymbol{q}' - \boldsymbol{q}'\boldsymbol{q}$. Then it is known that $\mathbb{H}$ endowed with $[\ ,\ ]$ is a Lie algebra over $\mathbb{R}$. Moreover, it is easily verified

(1) $[\mathbf{1}, \boldsymbol{q}] = [\boldsymbol{q}, \mathbf{1}] = 0, \qquad \boldsymbol{q} \in \mathbb{H}$;
(2) $[\boldsymbol{i}, \boldsymbol{i}] = [\boldsymbol{j}, \boldsymbol{j}] = [\boldsymbol{k}, \boldsymbol{k}] = 0$;
(3) $[\boldsymbol{i}, \boldsymbol{j}] = -[\boldsymbol{j}, \boldsymbol{i}] = 2\boldsymbol{k}$, $[\boldsymbol{j}, \boldsymbol{k}] = -[\boldsymbol{k}, \boldsymbol{j}] = 2\boldsymbol{i}$, $[\boldsymbol{k}, \boldsymbol{i}] = -[\boldsymbol{i}, \boldsymbol{k}] = 2\boldsymbol{j}$.

Consequently, we have

$$[\mathbb{C}, \mathbb{C}] = 0, \quad [\mathbb{D}, \mathbb{D}] = \mathbb{R}\boldsymbol{i}, \quad [\mathbb{C}, \mathbb{D}] = [\mathbb{D}, \mathbb{C}] = \mathbb{D}.$$

Let n be a positive integer. By $M(n; \mathbb{H})$ we denote the space of square matrices of degree n over the field $\mathbb{H}$. We will regard $M(n; \mathbb{H})$ as a $4n^2$-dimensional vector space over $\mathbb{R}$. Define a bracket in $M(n; \mathbb{H})$ by $[X, Y] = XY - YX$ $(X, Y \in M(n; \mathbb{H}))$. Then $M(n; \mathbb{H})$ endowed with $[\ ,\ ]$, which is a natural extension of the bracket $[\ ,\ ]$ defined in $\mathbb{H} = M(1; \mathbb{H})$, forms a Lie algebra over $\mathbb{R}$. For an element $X = \left(X_i^j\right)_{1\leq i,j\leq n} \in M(n;\mathbb{H})$ we mean by $\overline{X}$ the conjugate matrix $\overline{X} = \left(\overline{X_i^j}\right)_{1\leq i,j\leq n}$. Then we have $\overline{\overline{X}} = X$ and

$$ {}^t\overline{XY} = {}^t\overline{Y}\ {}^t\overline{X}, \qquad X, Y \in M(n;\mathbb{H}). $$

Now define a real bilinear form $\langle\ ,\ \rangle$ of $M(n;\mathbb{H})$ by

$$ \langle X, Y\rangle = \mathrm{Re}(\mathrm{Trace}(X\,{}^t\overline{Y})), \qquad X, Y \in M(n;\mathbb{H}). $$

It can be easily verified that $\langle\ ,\ \rangle$ is symmetric and positive definite on $M(n;\mathbb{H})$, i.e., $\langle\ ,\ \rangle$ is an inner product of $M(n;\mathbb{H})$. With this inner product $\langle\ ,\ \rangle$ we may regard $M(n;\mathbb{H})$ as the euclidean space $\mathbb{R}^{4n^2}$.

Let $Sp(n)$ denote the symplectic group of degree n, i.e., $Sp(n)$ is the subset of $M(n;\mathbb{H})$ consisting of all $g \in M(n;\mathbb{H})$ such that

$$ g\,{}^t\overline{g} = {}^t\overline{g}g = 1_n, $$

where 1_n is the identity matrix of order n. Let $\mathfrak{sp}(n)$ be the Lie algebra of $Sp(n)$. As is known, $\mathfrak{sp}(n)$ is a real subspace of $M(n;\mathbb{H})$ consisting of all $X \in M(n;\mathbb{H})$ such that

$$ X + {}^t\overline{X} = 0. $$

As is easily seen, $\dim \mathfrak{sp}(n) = 2n^2 + n$ and the inner product $\langle\ ,\ \rangle$ is invariant under the actions of $\mathrm{Ad}(Sp(n))$ and $\mathrm{ad}\,(\mathfrak{sp}(n))$:

$$ \begin{aligned} &\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle, && g \in Sp(n),\ X, Y \in M(n;\mathbb{H}); \\ &\langle \mathrm{ad}\,(Z)X, Y\rangle + \langle X, \mathrm{ad}\,(Z)Y\rangle = 0, && Z \in \mathfrak{sp}(n), X, Y \in M(n;\mathbb{H}). \end{aligned} $$

In the following we regard $\mathfrak{sp}(n)$ with the inner product $\langle\ ,\ \rangle$ as the euclidean space $\mathbb{R}^{2n^2+n}$. By $M(n;\mathbb{C})$ (resp. $M(n;\mathbb{D})$) we denote the subspace of $M(n;\mathbb{H})$ consisting of all matrices $X \in M(n;\mathbb{H})$ whose components are all contained in $\mathbb{C}$ (resp. $\mathbb{D}$). Then the unitary group $U(n)$ of degree n and its Lie algebra $\mathfrak{u}(n)$ are represented by $U(n) = Sp(n) \cap M(n;\mathbb{C})$ and $\mathfrak{u}(n) = \mathfrak{sp}(n) \cap M(n;\mathbb{C})$.

Lemma 10. *Let $\mathfrak{m}(n)$ be the space of symmetric matrices of degree n whose components are all contained in $\mathbb{D}$. Then the sum $\mathfrak{sp}(n) = \mathfrak{u}(n) + \mathfrak{m}(n)$ is an orthogonal direct sum with respect to $\langle\ ,\ \rangle$ and*

$$ [\mathfrak{u}(n), \mathfrak{u}(n)] \subset \mathfrak{u}(n);\ [\mathfrak{m}(n), \mathfrak{m}(n)] \subset \mathfrak{u}(n);\ [\mathfrak{u}(n), \mathfrak{m}(n)] \subset \mathfrak{m}(n). $$

In other words, $\mathfrak{sp}(n) = \mathfrak{u}(n) + \mathfrak{m}(n)$ gives the canonical decomposition of $\mathfrak{sp}(n)$ associated with the symmetric pair $(Sp(n), U(n))$.

Hereafter, we consider the symmetric space $M = Sp(n)/U(n)$. Identifying $\mathfrak{m}(n)$ with the tangent space $T_o(Sp(n)/U(n))$ at the origin o, we define an $Sp(n)$-invariant metric ν on $Sp(n)/U(n)$ by

$$\nu\big(X, Y\big) = \big\langle X, Y \big\rangle, \qquad X, Y \in \mathfrak{m}(n).$$

As is known, the Riemannian curvature R of type $(1,3)$ associated with ν is given as follows (see [14, Ch. XI]):

$$R(X, Y)Z = -[[X, Y], Z], \qquad X, Y, Z \in \mathfrak{m}(n).$$

Set $I_0 = (1/2)\boldsymbol{i}1_n\,(\in M(n;\mathbb{C}))$. Then I_0 is included in the center of $\mathfrak{u}(n)$ and satisfies

$$\begin{aligned} &\mathrm{ad}\,(I_0)X = \boldsymbol{i}X; \quad \mathrm{ad}\,(I_0)^2 X = -X, \qquad X \in \mathfrak{m}(n); \\ &\big\langle \mathrm{ad}\,(I_0)X, \mathrm{ad}\,(I_0)Y \big\rangle = \big\langle X, Y \big\rangle, \qquad X, Y \in \mathfrak{m}(n); \\ &\mathrm{Ad}(a) \cdot \mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)} = \mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)} \cdot \mathrm{Ad}(a), \qquad a \in U(n). \end{aligned}$$

Thus, it is easy to see that $\mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)}$ can be extended to an $Sp(n)$-invariant almost complex structure I. Thus the symmetric space $Sp(n)/U(n)$ endowed with the Riemannian metric ν and the almost complex structure I becomes a Hermitian symmetric space of compact type.

6. The Gauss equation on $Sp(n)/U(n)$

In this section, we consider the Gauss equation (2.1) at o modeled on the space $\mathfrak{u}(n)$, which is written in the form

$$\big\langle [[X, Y], Z], W \big\rangle = \big\langle \boldsymbol{\Psi}(X, Z), \boldsymbol{\Psi}(Y, W) \big\rangle - \big\langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Y, Z) \big\rangle, \tag{6.1}$$

where $\boldsymbol{\Psi} \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ and $X, Y, Z, W \in \mathfrak{m}(n)$. The inner product of $\mathfrak{u}(n)$ is taken to be the restriction of $\langle\ ,\ \rangle$ to the subspace $\mathfrak{u}(n)\,(\subset M(n;\mathbb{H}))$. Notations are the same in the previous sections. For simplicity, we set $\mathcal{G}(n) = \mathcal{G}_o(Sp(n)/U(n), \mathfrak{u}(n))$. In the following we will prove

Theorem 11. *Assume that $n \geq 2$. Then any solution $\boldsymbol{\Psi} \in \mathcal{G}(n)$ is Hermitian, i.e.,*

$$\boldsymbol{\Psi}(IX, IY) = \boldsymbol{\Psi}(X, Y), \qquad X, Y \in \mathfrak{m}(n).$$

If Theorem 11 is true, then by Proposition 9 we conclude that at each $p \in Sp(n)/U(n)$, $\mathcal{G}_p(Sp(n)/U(n), \boldsymbol{N})$ is EOS when $n \geq 2$ and $\dim \boldsymbol{N} = n^2$. This shows that $Sp(n)/U(n)$ ($n \geq 2$) is formally rigid in codimension n^2, proving Theorem 4.

For the proof of Theorem 11 we make several preparations. Let $\boldsymbol{\Psi} \in \mathcal{G}(n)$. For each $X \in \mathfrak{m}(n)$ we define a linear map $\boldsymbol{\Psi}_X$ of $\mathfrak{m}(n)$ to $\mathfrak{u}(n)$ by $\boldsymbol{\Psi}_X(Y) = \boldsymbol{\Psi}(X, Y)$ ($Y \in \mathfrak{m}(n)$). By $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ we denote the kernel of $\boldsymbol{\Psi}_X$. Then we have

Proposition 12. *Let* $\mathbf{\Psi} \in \mathcal{G}(n)$ *and* $X \in \mathfrak{m}(n)$. *Then*

(1) $\dim \boldsymbol{K}_{\mathbf{\Psi}}(X) \geq n$.

(2) $[[\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], X] = 0$.

Proof. (1) is clear from $\dim \boldsymbol{K}_{\mathbf{\Psi}}(X) \geq \dim \mathfrak{m}(n) - \dim \mathfrak{u}(n) = n$. Let $Y,\ Z \in \boldsymbol{K}_{\mathbf{\Psi}}(X)$ and let W be an arbitrary element of $\mathfrak{m}(n)$. Then by the Gauss equation (6.1) we have

$$\big\langle [[Y,Z],X],W \big\rangle = \big\langle \mathbf{\Psi}(Y,X), \mathbf{\Psi}(Z,W) \big\rangle - \big\langle \mathbf{\Psi}(Y,W), \mathbf{\Psi}(Z,X) \big\rangle = 0.$$

Since W is an arbitrary element, we have $[[Y,Z],X] = 0$, proving (2). □

Let $X \in \mathfrak{m}(n)$. We define a subspace $C(X) \subset \mathfrak{m}(n)$ by $C(X) = \{Y \in \mathfrak{m}(n) \mid [X,Y] = 0\}$. Then we have $\dim C(X) \geq \operatorname{rank}(Sp(n)/U(n)) = n$. We say an element $X \in \mathfrak{m}(n)$ is *regular* if $\dim C(X) = n$. It is obvious that for a regular element $X \in \mathfrak{m}(n)$, $C(X)$ is a unique maximal abelian subspace of $\mathfrak{m}(n)$ containing X. More strongly, since $\operatorname{rank}(Sp(n)) = n$, $C(X)$ is a unique maximal abelian subalgebra of $\mathfrak{sp}(n)$ containing X. We note that the set of regular elements forms an open dense subset of $\mathfrak{m}(n)$ and that any maximal abelian subspace $\mathfrak{a}$ contains regular elements as an open dense subset with respect to the induced topology of $\mathfrak{a}$.

Proposition 13. *Let* $\mathbf{\Psi} \in \mathcal{G}(n)$ *and* $X \in \mathfrak{m}(n)$.

(1) *If* $X \neq 0$, *then* $X \notin \boldsymbol{K}_{\mathbf{\Psi}}(X)$.

(2) *If* X *is regular, then* $\boldsymbol{K}_{\mathbf{\Psi}}(X)$ *is a maximal abelian subspace of* $\mathfrak{m}(n)$. *Moreover,*

$$\mathbf{\Psi}(\boldsymbol{K}_{\mathbf{\Psi}}(X), C(X)) = 0. \tag{6.2}$$

(3) *If* X *is not regular, then* $\dim \boldsymbol{K}_{\mathbf{\Psi}}(X) \geq \dim C(X)\ (> n)$.

Proof. Let $X \in \mathfrak{m}(n)$. Putting $Y = W = IX$ and $Z = X$ into (6.1), we have

$$\big\langle [[X,IX],X],IX \big\rangle = \big\langle \mathbf{\Psi}(X,X), \mathbf{\Psi}(IX,IX) \big\rangle - \big\langle \mathbf{\Psi}(X,IX), \mathbf{\Psi}(IX,X) \big\rangle.$$

Assume that $X \in \boldsymbol{K}_{\mathbf{\Psi}}(X)$, i.e., $\mathbf{\Psi}(X,X) = 0$. Then the right side of the above equality becomes $-|\mathbf{\Psi}(X,IX)|^2 \leq 0$, where $|\cdot|$ means the norm determined by $\langle\ ,\ \rangle$. On the other hand, since $Sp(n)/U(n)$ has positive holomorphic sectional curvature, the left side of the above equality becomes > 0 when $X \neq 0$. This is a contradiction. Hence we have $X \notin \boldsymbol{K}_{\mathbf{\Psi}}(X)$ if $X \neq 0$.

Next we show (2). Assume that $X \in \mathfrak{m}(n)$ is regular. Since $[\mathfrak{m}(n), \mathfrak{m}(n)] \subset \mathfrak{u}(n)$ and $[\mathfrak{u}(n), \mathfrak{m}(n)] \subset \mathfrak{m}(n)$, it follows that $[[\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], C(X)] \subset \mathfrak{m}(n)$. In view of (2) of Proposition 12, we easily get $[X, [[\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], C(X)]] = 0$. Consequently, $[[\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], C(X)] \subset C(X)$. Since $C(X)$ is an abelian subspace, we have

$$\big\langle [[\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], C(X)], C(X) \big\rangle = \big\langle [\boldsymbol{K}_{\mathbf{\Psi}}(X), \boldsymbol{K}_{\mathbf{\Psi}}(X)], [C(X), C(X)] \big\rangle = 0.$$

This implies that $[[\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)], C(X)] = 0$. Let $W \in [\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)]$. Then the sum $C(X) + \mathbb{R}W$ forms an abelian subalgebra of $\mathfrak{sp}(n)$. Since $C(X)$ is a unique maximal abelian subalgebra of $\mathfrak{sp}(n)$ containing X, we have $C(X) = C(X) + \mathbb{R}W$. Therefore $W \in C(X)$. However, since $W \in \mathfrak{u}(n)$ and $C(X) \subset \mathfrak{m}(n)$, we have $W = 0$. This proves $[\boldsymbol{K}_{\boldsymbol{\Psi}}(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)] = 0$, i.e., $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ is an abelian subspace of $\mathfrak{m}(n)$. Since $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(X) \geq n$, it follows that $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(X) = n$ and hence $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ is a maximal abelian subspace of $\mathfrak{m}(n)$.

Now take a regular element $Y \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X)$. Then it follows that $\boldsymbol{\Psi}(Y, C(X)) = 0$. In fact, as we have shown, $\boldsymbol{K}_{\boldsymbol{\Psi}}(Y)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ and satisfies $\boldsymbol{\Psi}_Y(X) = \boldsymbol{\Psi}(X, Y) = \boldsymbol{\Psi}_X(Y) = 0$, i.e., $X \in \boldsymbol{K}_{\boldsymbol{\Psi}}(Y)$. Since $C(X)$ is a unique maximal abelian subspace containing X, we have $\boldsymbol{K}_{\boldsymbol{\Psi}}(Y) = C(X)$, which proves $\boldsymbol{\Psi}(Y, C(X)) = 0$. Note that regular elements of $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ form an open dense subset of $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$. Therefore by continuity of $\boldsymbol{\Psi}$ we have $\boldsymbol{\Psi}(Y', C(X)) = 0$ for any $Y' \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X)$, i.e., $\boldsymbol{\Psi}(\boldsymbol{K}_{\boldsymbol{\Psi}}(X), C(X)) = 0$, completing the proof of (2).

Finally, assume that $X \in \mathfrak{m}(n)$ is not regular. Let $\mathfrak{a}$ be a maximal abelian subspace containing X. Since X is not regular, we have $C(X) \supsetneqq \mathfrak{a}$. Take a regular element $H \in \mathfrak{a}$. Then, since $\boldsymbol{K}_{\boldsymbol{\Psi}}(H)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see (2)), we can take a regular element $Z \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H)$. We now show that the image of $C(X)$ via the map $\boldsymbol{\Psi}_Z$ is isomorphic to the quotient $C(X)/\mathfrak{a}$, i.e., $\boldsymbol{\Psi}_Z(C(X)) \cong C(X)/\mathfrak{a}$. In fact, since $C(H) = \mathfrak{a}$, it follows that

$$\boldsymbol{\Psi}_Z(\mathfrak{a}) = \boldsymbol{\Psi}(Z, \mathfrak{a}) \subset \boldsymbol{\Psi}(\boldsymbol{K}_{\boldsymbol{\Psi}}(H), C(H)) = 0,$$

i.e., $\boldsymbol{K}_{\boldsymbol{\Psi}}(Z) \supset \mathfrak{a}$ (see (6.2)). Since $\boldsymbol{K}_{\boldsymbol{\Psi}}(Z)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see (2)), we have $\boldsymbol{K}_{\boldsymbol{\Psi}}(Z) = \mathfrak{a}$, proving our assertion.

Now let $Y \in C(X)$. Then by the Gauss equation (6.1) we have

$$\big\langle [[X, Y], Z], W \big\rangle = \big\langle \boldsymbol{\Psi}(X, Z), \boldsymbol{\Psi}(Y, W) \big\rangle - \big\langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Y, Z) \big\rangle, \quad W \in \mathfrak{m}(n).$$

Since $[X, Y] = 0$ and $\boldsymbol{\Psi}(X, Z) = \boldsymbol{\Psi}_Z(X) = 0$, we have

$$\big\langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Y, Z) \big\rangle = \big\langle \boldsymbol{\Psi}_X(W), \boldsymbol{\Psi}_Z(Y) \big\rangle = 0.$$

Consequently, the subspace $\boldsymbol{\Psi}_X(\mathfrak{m}(n))$ of $\mathfrak{u}(n)$ is perpendicular to the subspace $\boldsymbol{\Psi}_Z(C(X))$. Hence we have $\dim \boldsymbol{\Psi}_X(\mathfrak{m}(n)) \leq \dim \mathfrak{u}(n) - \dim \boldsymbol{\Psi}_Z(C(X))$. On the other hand, since $\boldsymbol{\Psi}_Z(C(X)) \cong C(X)/\mathfrak{a}$, it follows that $\dim \boldsymbol{\Psi}_Z(C(X)) = \dim C(X) - n$. Therefore,

$$\begin{aligned} \dim \boldsymbol{K}_{\boldsymbol{\Psi}}(X) &= \dim \mathfrak{m}(n) - \dim \boldsymbol{\Psi}_X(\mathfrak{m}(n)) \\ &\geq \dim \mathfrak{m}(n) - (\dim \mathfrak{u}(n) - \dim \boldsymbol{\Psi}_Z(C(X))) \\ &= (\dim \mathfrak{m}(n) - \dim \mathfrak{u}(n)) + \dim \boldsymbol{\Psi}_Z(C(X)) \\ &= n + (\dim C(X) - n) \\ &= \dim C(X), \end{aligned}$$

completing the proof of (3). □

Proposition 14. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$. *Let* $\mathfrak{a}$ *be a maximal abelian subspace of* $\mathfrak{m}(n)$. *Then:*

(1) *There exists a unique maximal abelian subspace* $\mathfrak{a}'$ *of* $\mathfrak{m}(n)$ *such that*

$$\boldsymbol{\Psi}(\mathfrak{a}, \mathfrak{a}') = 0. \tag{6.3}$$

(2) *Let* $\{H_1, \cdots, H_n\}$ *be a basis of* $\mathfrak{a}$. *Then the maximal abelian subspace* $\mathfrak{a}'$ *stated in* (1) *can be written as*

$$\mathfrak{a}' = \bigcap_{i=1}^{n} \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i).$$

Proof. First we prove the existence of $\mathfrak{a}'$ satisfying (6.3). Take a regular element $X \in \mathfrak{a}$ and set $\mathfrak{a}' = K_{\boldsymbol{\Psi}}(X)$. Then, we know that $\mathfrak{a}'$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see Proposition 13 (2)). Since $C(X) = \mathfrak{a}$, by (6.2) we obtain $\boldsymbol{\Psi}(\mathfrak{a}, \mathfrak{a}') = \boldsymbol{\Psi}(C(X), \boldsymbol{K}_{\boldsymbol{\Psi}}(X)) = 0$. Next, we prove the uniqueness of $\mathfrak{a}'$. Let $\mathfrak{a}'$ be a maximal abelian subspace satisfying (6.3). Take an arbitrary regular element X contained in $\mathfrak{a}$. Then by (6.3) it is clear that $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) \supset \mathfrak{a}'$. Since $\boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ is a maximal abelian subspace of $\mathfrak{m}(n)$, we have $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) = \mathfrak{a}'$. This proves the uniqueness of $\mathfrak{a}'$.

Let $\{H_1, \cdots, H_n\}$ be a basis of $\mathfrak{a}$. Then by (6.3) we have $\boldsymbol{\Psi}(H_i, \mathfrak{a}') = 0$ and hence $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_i) \supset \mathfrak{a}'$. Therefore, $\mathfrak{a}' \subset \bigcap_{i=1}^{n} \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i)$. On the other hand, by linearity of $\boldsymbol{\Psi}$, we have $\boldsymbol{K}_{\boldsymbol{\Psi}}(X) \supset \bigcap_{i=1}^{n} \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i)$ for any $X \in \mathfrak{a}$. In particular, if X is regular, then we have $\mathfrak{a}' = \boldsymbol{K}_{\boldsymbol{\Psi}}(X)$ and hence $\mathfrak{a}' \supset \bigcap_{i=1}^{n} \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i)$. This completes the proof of (2). □

Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ and $a \in U(n)$. Define an element $\boldsymbol{\Psi}^a \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ by

$$\boldsymbol{\Psi}^a(X, Y) = \boldsymbol{\Psi}(\mathrm{Ad}(a^{-1})X, \mathrm{Ad}(a^{-1})Y), \quad X, Y \in \mathfrak{m}(n).$$

Then, since $\mathrm{Ad}(a^{-1})$ preserves the curvature, we have $\boldsymbol{\Psi}^a \in \mathcal{G}(n)$ if and only if $\boldsymbol{\Psi} \in \mathcal{G}(n)$. We can easily show the following

Lemma 15. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$ *and* $a \in U(n)$. *Then*

$$\boldsymbol{K}_{\boldsymbol{\Psi}^a}(\mathrm{Ad}(a)X) = \mathrm{Ad}(a)(\boldsymbol{K}_{\boldsymbol{\Psi}}(X)), \quad X \in \mathfrak{m}(n).$$

Proof. The proof is obtained by the following

$$\begin{aligned} Y \in \boldsymbol{K}_{\boldsymbol{\Psi}^a}(\mathrm{Ad}(a)X) &\Longleftrightarrow \boldsymbol{\Psi}^a(\mathrm{Ad}(a)X, Y) = 0 \\ &\Longleftrightarrow \boldsymbol{\Psi}(X, \mathrm{Ad}(a^{-1})Y) = 0 \\ &\Longleftrightarrow \mathrm{Ad}(a^{-1})Y \in \boldsymbol{K}_{\boldsymbol{\Psi}}(X) \\ &\Longleftrightarrow Y \in \mathrm{Ad}(a)(\boldsymbol{K}_{\boldsymbol{\Psi}}(X)). \end{aligned}$$

□

Finally, we state Theorem 11 in a different form, which is somewhat easy to prove. Let E_{ij} $(1 \leq i, j \leq n)$ denote the matrix $\left(\delta_{is}\delta_{jt}\right)_{1\leq s,t\leq n} \in M(n;\mathbb{H})$. Then it is easily seen that the sum $\mathfrak{a}_0 = \sum_{i=1}^n \mathbb{R}\boldsymbol{j}E_{ii}$ forms a maximal abelian subspace of $\mathfrak{m}(n)$. Now consider the following:

Proposition 16. *Assume that* $n \geq 2$. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$. *Then*

$$\boldsymbol{\Psi}(\mathfrak{a}_0, I\mathfrak{a}_0) = 0.$$

We now show that Proposition 16 implies Theorem 11. Assume that Proposition 16 is true. Under this setting we will show that any element $\boldsymbol{\Psi} \in \mathcal{G}(n)$ is Hermitian. Let X be an arbitrary element of $\mathfrak{m}(n)$. As is known, there is an element $a \in U(n)$ such that $H = \mathrm{Ad}(a)X \in \mathfrak{a}_0$. By Proposition 16 we have $\boldsymbol{\Psi}^a(H, IH) = 0$, because $\boldsymbol{\Psi}^a \in \mathcal{G}(n)$. Noticing the relation $\mathrm{Ad}(a^{-1})I = I\mathrm{Ad}(a^{-1})$, we have

$$0 = \boldsymbol{\Psi}^a(H, IH) = \boldsymbol{\Psi}(\mathrm{Ad}(a^{-1})H, \mathrm{Ad}(a^{-1})IH) = \boldsymbol{\Psi}(X, IX).$$

Consequently, for any $X \in \mathfrak{m}(n)$ we have $\boldsymbol{\Psi}(X, IX) = 0$, which means that $\boldsymbol{\Psi}$ is Hermitian. Thus we get Theorem 11.

7. Proof of Proposition 16

In this section we will prove Proposition 16. Let n' be a non-negative integer such that $n' < n$. By the assignment

$$\mathfrak{m}(n') \ni X \longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}(n)$$

we may regard $\mathfrak{m}(n')$ as a subspace of $\mathfrak{m}(n)$. In the special case $n' = n-1$ we have the direct sum

$$\mathfrak{m}(n) = \mathfrak{m}(n-1) + \mathbb{D}E_{nn} + \sum_{i=1}^{n-1} \mathbb{D}(E_{in} + E_{ni}). \tag{7.1}$$

For simplicity we set $H_i = \boldsymbol{j}E_{ii}$ $(i = 1, \ldots, n)$. Then we have $IH_i = \boldsymbol{i}H_i = \boldsymbol{k}E_{ii}$. Consequently, we have

$$\mathbb{C}H_i = \mathbb{R}H_i + \mathbb{R}IH_i = \mathbb{D}E_{ii}, \qquad i = 1, \ldots, n.$$

As in the previous section we set $\mathfrak{a}_0 = \sum_{i=1}^n \mathbb{R}H_i$. In the following we will prove $\boldsymbol{\Psi}(\mathfrak{a}_0, I\mathfrak{a}_0) = 0$ for any $\boldsymbol{\Psi} \in \mathcal{G}(n)$. First we show

Lemma 17. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$. *Then there exists a real number* $a \in \mathbb{R}$ *such that*

$$\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) = \mathfrak{m}(n-1) + \mathbb{R}(IH_n - aH_n) \qquad (\textit{direct sum}). \tag{7.2}$$

Accordingly, $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) = \dim \mathfrak{m}(n-1) + 1$.

Proof. First assume that the following inclusion holds:

$$\boldsymbol{K}_{\Psi}(H_n) \subset \mathfrak{m}(n-1) + \mathbb{C}H_n. \tag{7.3}$$

Then, since $H_n \notin \boldsymbol{K}_{\Psi}(H_n)$, we have $\dim \boldsymbol{K}_{\Psi}(H_n) \leq \dim \mathfrak{m}(n-1)+1$. On the other hand, by a simple calculation we can verify that $C(H_n) = \mathfrak{m}(n-1) + \mathbb{R}H_n$. On account of the relation $\dim \boldsymbol{K}_{\Psi}(H_n) \geq \dim C(H_n)$ we have $\dim \boldsymbol{K}_{\Psi}(H_n) = \dim \mathfrak{m}(n-1)+1$. Moreover, we have

$$\boldsymbol{K}_{\Psi}(H_n) + \mathbb{R}H_n = \mathfrak{m}(n-1) + \mathbb{C}H_n. \tag{7.4}$$

Now we show $\mathfrak{m}(n-1) \subset \boldsymbol{K}_{\Psi}(H_n)$. By (7.4) it is known that there is a real number a such that $IH_n - aH_n \in \boldsymbol{K}_{\Psi}(H_n)$. Similarly, for any $X \in \mathfrak{m}(n-1)$, there is a real number $b \in \mathbb{R}$ such that $X - bH_n \in \boldsymbol{K}_{\Psi}(H_n)$. Consider the equality $[[IH_n - aH_n, X - bH_n], H_n] = 0$. By a direct calculation we have $[H_n, X] = [IH_n, X] = 0$ and $[[IH_n, H_n], H_n] = -4IH_n$. Consequently, $[[IH_n - aH_n, X - bH_n], H_n] = 4bIH_n = 0$, implying $b = 0$. Hence we have $X \in \boldsymbol{K}_{\Psi}(H_n)$, i.e., $\mathfrak{m}(n-1) \subset \boldsymbol{K}_{\Psi}(H_n)$. Thus if (7.3) is true, then we obtain the lemma.

Now we suppose that (7.3) is not true, i.e., $\boldsymbol{K}_{\Psi}(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$. Let r and s be non-negative integers. By $M(r,s;\mathbb{D})$ we denote the space of $\mathbb{D}$-valued $r \times s$-matrices. As is easily seen, each element $X \in \mathfrak{m}(n)$ can be written in the form

$$X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix}, \qquad X' \in \mathfrak{m}(n-1), \xi \in M(n-1,1;\mathbb{D}), \boldsymbol{x} \in \mathbb{D}.$$

Under our assumption $\boldsymbol{K}_{\Psi}(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$ we know that there is an element $X = \left(\begin{smallmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{smallmatrix}\right) \in \boldsymbol{K}_{\Psi}(H_n)$ such that $\xi \neq 0$. Let $\varphi\colon \mathfrak{m}(n) \longrightarrow M(n-1,1;\mathbb{D})$ be the natural projection defined by $\varphi(\left(\begin{smallmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{smallmatrix}\right)) = \xi$. By $\varphi(\boldsymbol{K}_{\Psi}(H_n))$ we denote the image of $\boldsymbol{K}_{\Psi}(H_n)$ by φ. Then we have $\varphi(\boldsymbol{K}_{\Psi}(H_n)) \neq 0$. As is easily seen, from the right multiplication $\xi \longmapsto \xi c$ of $c \in \mathbb{C}$, $M(n-1,1;\mathbb{D})$ may be regarded as a complex vector space with $\dim_{\mathbb{C}} M(n-1,1;\mathbb{D}) = n-1$. By $\varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}}$ we mean the complex subspace of $M(n-1,1;\mathbb{D})$ generated by $\varphi(\boldsymbol{K}_{\Psi}(H_n))$, i.e., $\varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}} = \varphi(\boldsymbol{K}_{\Psi}(H_n)) + \varphi(\boldsymbol{K}_{\Psi}(H_n))\boldsymbol{i}$. Set $s = \dim_{\mathbb{C}} \varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}}$. Then, clearly we have $1 \leq s \leq n-1$, $\dim \varphi(\boldsymbol{K}_{\Psi}(H_n)) \leq 2s$ and

$$\dim \boldsymbol{K}_{\Psi}(H_n) = \dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\Psi}(H_n)) + \dim \varphi(\boldsymbol{K}_{\Psi}(H_n)). \tag{7.5}$$

Now, let us show

$$\dim(\mathfrak{m}(n-1) \cap \boldsymbol{K}_{\Psi}(H_n)) \leq (n-s-1)(n-s). \tag{7.6}$$

Let $\mathfrak{m}(n-1)'$ be the subspace of $\mathfrak{m}(n-1)$ consisting of all $Y' \in \mathfrak{m}(n-1)$ satisfying $Y = \left(\begin{smallmatrix} Y' & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\Psi}(H_n)$. To show (7.6) it suffices to prove $\dim \mathfrak{m}(n-1)' \leq (n-s-1)(n-s)$. For the proof we prepare the following formula:

$$[[X,Y],H_n] = \begin{pmatrix} 0 & (\xi\boldsymbol{y} - Y'\xi)\boldsymbol{j} \\ -\boldsymbol{j}({}^t\xi Y' - \boldsymbol{y}\,{}^t\xi) & [[\boldsymbol{x},\boldsymbol{y}],\boldsymbol{j}] \end{pmatrix}, \tag{7.7}$$

where $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix} \in \mathfrak{m}(n)$ and $Y = \begin{pmatrix} Y' & 0 \\ 0 & \boldsymbol{y} \end{pmatrix} \in \mathfrak{m}(n-1) + \mathbb{C}H_n$. This formula can be easily obtained by a simple calculation. Utilizing (7.7), we show (7.6). Let $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix} \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$ and $Y = \begin{pmatrix} Y' & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Since $[[X, Y], H_n] = 0$, by (7.7) it follows that $Y'\xi = 0$. We note that this equality holds for any $\xi \in \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))$ and $Y' \in \mathfrak{m}(n-1)'$. Since $Y'(\xi \boldsymbol{i}) = (Y'\xi)\boldsymbol{i}$, we have

$$Y'\xi = 0, \qquad \forall Y' \in \mathfrak{m}(n-1)',\ \forall \xi \in \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))^{\mathbb{C}}. \tag{7.8}$$

Select a basis $\{\eta_1, \cdots, \eta_{n-s-1}, \xi_1, \cdots, \xi_s\}$ of the complex vector space $M(n-1, 1; \mathbb{D})$ such that $\{\xi_1, \cdots, \xi_s\}$ forms a basis of $\varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))^{\mathbb{C}}$. Define a matrix $U \in M(n-1; \mathbb{H})$ by $U = (\eta_1, \cdots, \eta_{n-s-1}, \xi_1, \cdots, \xi_s)$. Let $Y' \in \mathfrak{m}(n-1)'$. Since $Y'\xi_1 = \cdots = Y'\xi_s = 0$ and ${}^t\xi_1 Y' = \cdots = {}^t\xi_s Y' = 0$, we have ${}^tU \cdot Y' \cdot U \in \mathfrak{m}(n-s-1)$. This means that ${}^tU \cdot \mathfrak{m}(n-1)' \cdot U \subset \mathfrak{m}(n-s-1)$. Since U is a non-singular matrix, we have $\dim_{\mathbb{R}} \mathfrak{m}(n-1)' \leq \dim_{\mathbb{R}} \mathfrak{m}(n-s-1) = (n-s-1)(n-s)$, proving the desired inequality (7.6).

Next we consider the intersection $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Since $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \supset \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$, the following two cases are possible:

(i) $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) = \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$.
(ii) $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \supsetneq \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$.

In the case (i), we have $\dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) \leq (n-s-1)(n-s)$. Since $\dim \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) \leq 2s$, by (7.5) we have

$$\begin{aligned} \dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) &\leq (n-s-1)(n-s) + 2s \\ &= s^2 - (2n-3)s + n(n-1). \end{aligned} \tag{7.9}$$

Since $1 \leq s \leq n-1$, the right side of (7.9) attains its maximum when $s = 1$. Consequently, we have $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \leq 4 - 2n + n(n-1) < 1 + \dim \mathfrak{m}(n-1)$, because $\dim \mathfrak{m}(n-1) = n(n-1)$ and $n \geq 2$. This contradicts the fact $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \geq 1 + \dim \mathfrak{m}(n-1)$. Therefore we know that the case (i) is impossible.

Next we show the case (ii) is also impossible. Let $Y = \begin{pmatrix} Y' & 0 \\ 0 & \boldsymbol{y} \end{pmatrix}$ be an element of $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$ satisfying $\boldsymbol{y} \neq 0$. Let $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix}$ be an arbitrary element of $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Then, since $[[X, Y], H_n] = 0$, we obtain by (7.7)

$$[[\boldsymbol{x}, \boldsymbol{y}], \boldsymbol{j}] = 0; \qquad \xi \boldsymbol{y} - Y'\xi = 0. \tag{7.10}$$

From the first equality in (7.10) we have $\boldsymbol{x} \in \mathbb{R}\boldsymbol{y}$. In fact, since $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}$, we have $[\boldsymbol{x}, \boldsymbol{y}] \in \mathbb{R}\boldsymbol{i}$. However, since $[\boldsymbol{i}, \boldsymbol{j}] = 2\boldsymbol{k} \neq 0$, we have $[\boldsymbol{x}, \boldsymbol{y}] = 0$, implying $\boldsymbol{x} \in \mathbb{R}\boldsymbol{y}$. This fact means that for any element $X \in (\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$ there is a real number c such that $X - cY \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Consequently, we have $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \subset$

$\mathbb{R}Y + (\mathfrak{m}(n-1) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))$ and hence

$$\dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) \leq 1 + (n-s-1)(n-s). \tag{7.11}$$

Moreover, in this case we have $\dim \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) = s$. In fact, we have

$$\varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))^{\mathbb{C}} = \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) + \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))\boldsymbol{i} \quad \text{(direct sum)}. \tag{7.12}$$

It is easily seen that to show (7.12) it suffices to prove $\varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) \cap \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))\boldsymbol{i} = 0$. Assume that $\xi \in \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))$ satisfies $\xi\boldsymbol{i} \in \varphi(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n))$. Take elements X, $X_1 \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$ such that $\varphi(X) = \xi$, $\varphi(X_1) = \xi\boldsymbol{i}$. Then, since $[[X,Y],H_n] = [[X_1,Y],H_n] = 0$, by the second equality in (7.10) we have $\xi\boldsymbol{y} - Y'\xi = 0$ and $(\xi\boldsymbol{i})\boldsymbol{y} - Y'(\xi\boldsymbol{i}) = 0$. Since $\boldsymbol{i}\boldsymbol{y} = -\boldsymbol{y}\boldsymbol{i}$, the last equality becomes $(\xi\boldsymbol{i})\boldsymbol{y} - Y'(\xi\boldsymbol{i}) = -(\xi\boldsymbol{y} + Y'\xi)\boldsymbol{i} = 0$. Consequently, $\xi\boldsymbol{y} + Y'\xi = 0$, showing $\xi\boldsymbol{y} = 0$. Hence we get $\xi = 0$, because $\boldsymbol{y} \neq 0$.

Thus by (7.5) we have

$$\begin{aligned}\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) &\leq 1 + (n-s-1)(n-s) + s \\ &= s^2 - 2(n-1)s + \dim \mathfrak{m}(n-1) + 1.\end{aligned} \tag{7.13}$$

The right side of (7.13) attains its maximum when $s = 1$ and therefore $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \leq 4 - 2n + \dim \mathfrak{m}(n-1) < 1 + \dim \mathfrak{m}(n-1)$, which is also a contradiction.

Thus, assuming $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$, we meet a contradiction. Hence we have $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) \subset \mathfrak{m}(n-1) + \mathbb{C}H_n$, completing the proof of the lemma. □

In a similar manner we can prove the following lemma:

Lemma 18. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$. *Then*

$$\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i) = \dim \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_i) = \dim \mathfrak{m}(n-1) + 1, \quad i = 1, \dots, n.$$

Moreover there exist real numbers a' *and* $b \in \mathbb{R}$ *such that*

$$\boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n) = \mathfrak{m}(n-1) + \mathbb{R}(H_n - a'IH_n) \quad (\textit{direct sum});$$

$$\begin{aligned}\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) = \mathfrak{m}(n-2) &+ \sum_{i=1}^{n-2} \mathbb{D}(E_{in} + E_{ni}) + \mathbb{C}H_n \\ &+ \mathbb{R}(IH_{n-1} - bH_{n-1}) \quad (\textit{direct sum}).\end{aligned}$$

With the aid of Lemma 18 we can prove the refinement of Lemma 17.

Lemma 19. *Let* $\boldsymbol{\Psi} \in \mathcal{G}(n)$. *Then* $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) = \mathfrak{m}(n-1) + \mathbb{R}IH_n$ *(direct sum).*

Proof. Let $\boldsymbol{\Psi} \in \mathcal{G}(n)$. Take real numbers a, a' and $b \in \mathbb{R}$ stated in Lemma 17 and Lemma 18 and set $Y = IH_n - aH_n$, $Z = IH_{n-1} - bH_{n-1}$ and $W = H_n - a'IH_n$. Then clearly we have

$$Y = (\boldsymbol{k} - a\boldsymbol{j})E_{nn}, \ Z = (\boldsymbol{k} - b\boldsymbol{j})E_{n-1,n-1}, \ W = (\boldsymbol{j} - a'\boldsymbol{k})E_{nn}.$$

In the following we will show that $a = a' = b = 0$. If this can be done, the lemma follows immediately.

First we prove $a = b$. For this purpose we consider the space $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + H_n)$. By an easy calculation we can verify that $C(H_{n-1} + H_n) = \mathfrak{m}(n-2) + \mathbb{R}H_{n-1} + \mathbb{R}H_n + \mathbb{R}\boldsymbol{j}(E_{n-1,n} + E_{n,n-1})$. Therefore we have $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + H_n) \geq \dim C(H_{n-1} + H_n) = \dim \mathfrak{m}(n-2) + 3$. Since $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n) = \mathfrak{m}(n-2) + \mathbb{R}Y + \mathbb{R}Z$, it follows that $\dim(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)) = \dim \mathfrak{m}(n-2) + 2$. Consequently, we can take an element $X \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + H_n)$ such that $X \notin \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Write

$$X = \begin{pmatrix} X'' & \eta & \xi \\ {}^t\eta & \boldsymbol{y} & \boldsymbol{z} \\ {}^t\xi & \boldsymbol{z} & \boldsymbol{x} \end{pmatrix}, \tag{7.14}$$

where $X'' \in \mathfrak{m}(n-2)$, $\xi, \eta \in M(n-2, 1; \mathbb{D})$, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{D}$. Since $X, Y, Z \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}+H_n)$, we have $[[X, Y], H_{n-1} + H_n] = [[X, Z], H_{n-1} + H_n] = 0$. Set $\boldsymbol{h} = \boldsymbol{k} - a\boldsymbol{j}$ and $\boldsymbol{h}' = \boldsymbol{k} - b\boldsymbol{j}$. Then we have

$$\xi \boldsymbol{h}\boldsymbol{j} = \eta \boldsymbol{h}'\boldsymbol{j} = 0; \tag{7.15}$$

$$[[\boldsymbol{x}, \boldsymbol{h}], \boldsymbol{j}] = [[\boldsymbol{y}, \boldsymbol{h}'], \boldsymbol{j}] = 0; \tag{7.16}$$

$$[\boldsymbol{z}\boldsymbol{h}, \boldsymbol{j}] = [\boldsymbol{z}\boldsymbol{h}', \boldsymbol{j}] = 0. \tag{7.17}$$

By (7.15) and (7.16) we easily have $\xi = \eta = 0$, $\boldsymbol{x} \in \mathbb{R}\boldsymbol{h}$, and $\boldsymbol{y} \in \mathbb{R}\boldsymbol{h}'$. Thus, if $\boldsymbol{z} = 0$, then we have $X \in \mathfrak{m}(n-2) + \mathbb{R}Y + \mathbb{R}Z = \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. This contradicts the assumption $X \notin \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Hence $\boldsymbol{z} \neq 0$. Now consider (7.17). First note that $\boldsymbol{z}\boldsymbol{h}$, $\boldsymbol{z}\boldsymbol{h}' \in \mathbb{C} = \mathbb{R} + \mathbb{R}\boldsymbol{i}$. Since $[\boldsymbol{i}, \boldsymbol{j}] \neq 0$, $[\boldsymbol{z}\boldsymbol{h}, \boldsymbol{j}] = [\boldsymbol{z}\boldsymbol{h}', \boldsymbol{j}] = 0$ holds if and only if $\boldsymbol{z}\boldsymbol{h}$, $\boldsymbol{z}\boldsymbol{h}' \in \mathbb{R}$. Since $\boldsymbol{z} \neq 0$, we have $\boldsymbol{h} \in \mathbb{R}\boldsymbol{z}^{-1}$ and $\boldsymbol{h}' \in \mathbb{R}\boldsymbol{z}^{-1}$. Consequently, we have $\mathbb{R}\boldsymbol{h} = \mathbb{R}\boldsymbol{z}^{-1}$ and hence $\boldsymbol{h}' \in \mathbb{R}\boldsymbol{h} = \mathbb{R}(\boldsymbol{k} - a\boldsymbol{j})$. Therefore we have $a = b$, because $\boldsymbol{h}' = \boldsymbol{k} - b\boldsymbol{j}$.

Next we prove $a' = -a$. For this purpose we consider the space $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + IH_n)$. We can easily see that $C(H_{n-1} + IH_n) = \mathfrak{m}(n-2) + \mathbb{R}H_{n-1} + \mathbb{R}IH_n + \mathbb{R}(\boldsymbol{j} + \boldsymbol{k})(E_{n-1,n} + E_{n,n-1})$. Hence $\dim \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + IH_n) \geq \dim \mathfrak{m}(n-2) + 3$. Since $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n) = \mathfrak{m}(n-2) + \mathbb{R}Z + \mathbb{R}W$, it follows that $\dim(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n)) = \dim \mathfrak{m}(n-2) + 2$. Consequently, we can take an element $X \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + IH_n)$ such that $X \notin \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n)$. Since X, Z, $W \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1} + IH_n)$, we have $[[X, Z], H_{n-1} + IH_n] = [[X, W], H_{n-1} + IH_n] = 0$. Writing X in the form (7.14), we have

$$\xi \boldsymbol{h}''\boldsymbol{k} = \eta \boldsymbol{h}\boldsymbol{j} = 0; \tag{7.18}$$

$$[[\boldsymbol{x}, \boldsymbol{h}''], \boldsymbol{k}] = [[\boldsymbol{y}, \boldsymbol{h}], \boldsymbol{j}] = 0; \tag{7.19}$$

$$\boldsymbol{z}\boldsymbol{h}''\boldsymbol{k} - \boldsymbol{j}\boldsymbol{z}\boldsymbol{h}'' = \boldsymbol{h}\boldsymbol{z}\boldsymbol{k} - \boldsymbol{j}\boldsymbol{h}\boldsymbol{z} = 0, \tag{7.20}$$

where we set $\boldsymbol{h} = \boldsymbol{k} - a\boldsymbol{j}$ and $\boldsymbol{h}'' = \boldsymbol{j} - a'\boldsymbol{k}$. By (7.18) and (7.19) we have $\xi = \eta = 0$, $\boldsymbol{x} \in \mathbb{R}\boldsymbol{h}''$ and $\boldsymbol{y} \in \mathbb{R}\boldsymbol{h}$. Hence if $\boldsymbol{z} = 0$, then $X \in \mathfrak{m}(n-2) + \mathbb{R}Z + \mathbb{R}W = \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n)$. This contradicts the assumption $X \notin \boldsymbol{K}_{\boldsymbol{\Psi}}(H_{n-1}) \cap \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n)$. Hence $\boldsymbol{z} \neq 0$. Now consider (7.20). It is easily verified that $\boldsymbol{z}\boldsymbol{h}''\boldsymbol{k} - \boldsymbol{j}\boldsymbol{z}\boldsymbol{h}'' = \boldsymbol{h}\boldsymbol{z}\boldsymbol{k} - \boldsymbol{j}\boldsymbol{h}\boldsymbol{z} = 0$ holds when and only when $\boldsymbol{z}\boldsymbol{h}'' \in \mathbb{R}(1-\boldsymbol{i})$ and $\boldsymbol{h}\boldsymbol{z} \in \mathbb{R}(1-\boldsymbol{i})$. Since $\boldsymbol{z} \neq 0$, we have $\boldsymbol{h}'' \in \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$ and $\boldsymbol{h} \in \mathbb{R}(1-\boldsymbol{i})\boldsymbol{z}^{-1}$. Therefore, $(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) \in \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$. Consequently, $\mathbb{R}(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) = \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$ and hence $\boldsymbol{h}'' \in \mathbb{R}(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) = \mathbb{R}(\boldsymbol{j} + a\boldsymbol{k})$. Accordingly, we have $a' = -a$, because $\boldsymbol{h}'' = \boldsymbol{j} - a'\boldsymbol{k}$.

Finally, we prove $a = 0$. By the definition we have $Y = IH_n - aH_n \in \boldsymbol{K}_{\boldsymbol{\Psi}}(H_n)$. Moreover, by the above discussion we know $W = H_n + aIH_n \in \boldsymbol{K}_{\boldsymbol{\Psi}}(IH_n)$. Hence

$$\boldsymbol{\Psi}(H_n, IH_n - aH_n) = \boldsymbol{\Psi}(IH_n, H_n + aIH_n) = 0.$$

Consequently, we have $\boldsymbol{\Psi}(H_n, IH_n) = a\,\boldsymbol{\Psi}(H_n, H_n)$ and $\boldsymbol{\Psi}(H_n, IH_n) = -a\,\boldsymbol{\Psi}(IH_n, IH_n)$. If $a \neq 0$, then we have $\boldsymbol{\Psi}(IH_n, IH_n) = -\boldsymbol{\Psi}(H_n, H_n)$. Putting $X = Z = H_n$ and $Y = W = IH_n$ into (6.1), we have

$$\begin{aligned} \big\langle [[H_n, IH_n], H_n], IH_n \big\rangle &= \big\langle \boldsymbol{\Psi}(H_n, H_n), \boldsymbol{\Psi}(IH_n, IH_n) \big\rangle - \big\langle \boldsymbol{\Psi}(H_n, IH_n), \boldsymbol{\Psi}(IH_n, H_n) \big\rangle \\ &= -(1+a^2)\big\langle \boldsymbol{\Psi}(H_n, H_n), \boldsymbol{\Psi}(H_n, H_n) \big\rangle \\ &\leq 0. \end{aligned}$$

On the other hand, the left side is > 0, which is a contradiction. Thus we have $a = 0$, completing the proof of the lemma. □

We now complete the proof of Proposition 16.

Proof of Proposition 16. Assume that $n \geq 2$. Let $\boldsymbol{\Psi} \in \mathcal{G}(n)$. We will prove

$$\boldsymbol{K}_{\boldsymbol{\Psi}}(H_i) \supset I\mathfrak{a}_0, \qquad i = 1, \ldots, n. \tag{7.21}$$

Let i be an integer such that $1 \leq i \leq n$. If $i = n$, then (7.21) follows from Lemma 19. Now assume that $i < n$. Set $a = E_{ni} - E_{in} + \sum_{j=1, j\neq i}^{n-1} E_{jj}$. Then it is easy to see that $a \in SO(n)\,(\subset U(n))$, $\mathrm{Ad}(a)H_i = H_n$ and $\mathrm{Ad}(a)\mathfrak{a}_0 = \mathfrak{a}_0$. Consequently, by Lemma 15 we have $\boldsymbol{K}_{\boldsymbol{\Psi}^a}(H_n) = \mathrm{Ad}(a)(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_i))$. On the other hand, since $\boldsymbol{\Psi}^a \in \mathcal{G}(n)$, we have $\boldsymbol{K}_{\boldsymbol{\Psi}^a}(H_n) \supset I\mathfrak{a}_0$. This shows $\mathrm{Ad}(a)(\boldsymbol{K}_{\boldsymbol{\Psi}}(H_i)) \supset I\mathfrak{a}_0$ and hence $\boldsymbol{K}_{\boldsymbol{\Psi}}(H_i) \supset \mathrm{Ad}(a^{-1})(I\mathfrak{a}_0) = I\mathfrak{a}_0$. Consequently, we get (7.21), which implies $\bigcap_{i=1}^{n} \boldsymbol{K}_{\boldsymbol{\Psi}}(H_i) \supset I\mathfrak{a}_0$. Therefore, in view of Proposition 14, we have $\boldsymbol{\Psi}(\mathfrak{a}_0, I\mathfrak{a}_0) = 0$, proving the proposition. □

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