# Asymptotic forms of weakly increasing positive solutions of quasilinear ordinary differential equations

Ken-ichi Kamo and Hiroyuki Usami

Division of Mathematics, School of Medicine, Liberal Arts and Sciences, Sapporo Medical University
Department of Mathematics, Graduate School of Science, Hiroshima University

**Abstract**

Asymptotic forms are determined for positive solutions which are called weakly increasing solutions to quasilinear ordinary differential equations.

**Key words.** quasilinear ordinary differential equation, positive solution

## 1 Introduction

In the paper we consider the equations of the form

$$(|u'|^{\alpha-1}u')' + p(t)|u|^{\beta-1}u = 0 \tag{E}$$

under the following conditions:

($A_1$) $\alpha$ and $\beta$ are positive constants satisfying $\alpha \neq \beta$;
($A_2$) $p(t)$ is a $C^1-$function defined near $+\infty$ satisfying the asymptotic condition $p(t) \sim t^{-\sigma}$ for some $\sigma \in R$ as $t \to \infty$.

By condition ($A_2$) equation (E) can be rewritten in the form

$$(|u'|^{\alpha-1}u')' + t^{-\sigma}(1+\varepsilon(t))|u|^{\beta-1}u = 0, \tag{E}$$

where $\varepsilon(t) = t^{\sigma}p(t) - 1$ satisfies $\lim_{t\to\infty} \varepsilon(t) = 0$. Of course, here and in what follows the symbol "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$. Some of preparatory results for equation (E) are still valid for general equations than (E); so it is convenient to consider the auxiliary equation

$$(|u'|^{\alpha-1}u')' + q(t)|u|^{\beta-1}u = 0, \tag{1}$$

where we assume that $\alpha$ and $\beta$ satisfy condition ($A_1$) and $q \in C([t_0,\infty);(0,\infty))$. A function $u$ is defined to be a solution of equation (1) if $u \in C^1[t_1,\infty)$ and $|u'|^{\alpha-1}u' \in C^1[t_1,\infty)$ and it satisfies equation (1) on $[t_1,\infty)$ for sufficiently large $t_1$.

It is easily seen that all positive solutions $u(t)$ of (1) are classified into the following three types according as their asymptotic behavior as $t \to \infty$:

(I) (asymptotically linear solution):

$$u(t) \sim c_1 t \quad \text{for some constant} \quad c_1 > 0;$$

(II) (*weakly increasing solution*):

$$u'(t) \downarrow 0, \quad \text{and} \quad u(t) \uparrow \infty$$

(III) (asymptotically constant solution):

$$u(t) \sim c_1 \quad \text{for some constant} \quad c_1 > 0.$$

Concerning qualitative properties of positive solutions, the study of asymptotic behavior of asymptotically linear solutions and asymptotically constant solutions are rather easy, because their first approximations are given by definition. On the other hand, we can not easily know how the weakly increasing positive solutions behave except for the case of $\alpha = 1$ ([1, 3]).

In [3, Section 20], equation (E) with $\alpha = 1$ has been considered systematically, and asymptotic forms of weakly increasing positive solutions are given by means of the parameters $\beta$ and $\sigma$. When $\alpha \neq 1$, as far as the authors are aware, there are no works in which asymptotic forms of weakly increasing positive solutions are studied systematically.

Motivated by these facts in the paper we make an attempt to find out asymptotic forms of weakly increasing positive solutions of (E) for the general case $\alpha > 0$. Furthermore we will also establish more than obtained in [3] for the case of $\alpha > \beta$. In fact, some of our results are new even though $\alpha = 1$.

To gain an insight into our problem, we consider the typical equation

$$(|u'|^{\alpha-1}u')' + t^{-\sigma}|u|^{\beta-1}u = 0, \tag{E$_0$}$$

where $\sigma \in R$ is a constant. Note that equation (E) can be regarded as a perturbed equation of this equation. Equation (E$_0$) has a weakly increasing positive solution of the form $ct^{\rho}, (c > 0, 0 < \rho < 1)$ if and only if $\min\{\alpha, \beta\} + 1 < \sigma < \max\{\alpha, \beta\} + 1$. This solution is uniquely given by

$$u_0(t) = \hat{C}t^k, \tag{2}$$

where

$$k = \frac{\alpha - \sigma + 1}{\alpha - \beta} \in (0, 1), \quad \hat{C} = \{\alpha(1-k)k^{\alpha}\}^{\frac{1}{\beta-\alpha}}. \tag{3}$$

From this simple observation we can see that asymptotic forms of weakly increasing positive solutions of (1) may be strongly affected by that of the coefficient function $q(t)$. Furthermore we conjecture that weakly increasing positive solutions $u$ of (E) behave like $u_0(t)$ given by (2) and (3) if $|\varepsilon(t)|$ is sufficiently small at $\infty$.

We will show that the above conjecture is true in many cases. In fact, we can obtain the following theorems which are main results of the paper:

**Theorem 1** *Let $\alpha > \beta$.*

(i) *Suppose that $\beta + 1 < \sigma < \alpha + 1$. Then, every weakly increasing positive solution $u$ of* (E) *has the asymptotic form*

$$u(t) \sim u_0(t) \quad \textit{as} \quad t \to \infty,$$

*where* $u_0$ *is given by* (2) *and* (3).

(ii) *Suppose that* $\sigma = \alpha + 1$; *namely* $p(t) \sim t^{-\alpha-1}$ *as* $t \to \infty$. *Then, every weakly increasing positive solution* $u$ *of* (E) *has the asymptotic form*

$$u(t) \sim \alpha^{-\frac{1}{\alpha-\beta}} \left(\frac{\alpha}{\alpha-\beta}\right)^{-\frac{\alpha}{\alpha-\beta}} (\log t)^{\frac{\alpha}{\alpha-\beta}} \quad \textit{as} \quad t \to \infty.$$

**Theorem 2** *Let* $\alpha < \beta$. *Suppose that* $\alpha \le 1$ *and* $1/2 < k < 1 (\Leftrightarrow (\alpha+\beta+2)/2 < \sigma < \beta+1)$. *Suppose furthermore that either*

$$\int^{\infty} \frac{\varepsilon(t)^2}{t} dt < \infty \tag{4}$$

*or*

$$\int^{\infty} |\varepsilon'(t)| dt < \infty \tag{5}$$

*holds. Then, every weakly increasing positive solution* $u$ *of* (E) *has the asymptotic form*

$$u(t) \sim u_0(t) \quad \textit{as} \quad t \to \infty,$$

*where* $u_0$ *is given by* (2) *and* (3).

**Theorem 3** *Let* $\alpha < \beta$. *Suppose that* $\alpha \ge 1$ *and* $0 < k < 1/2 (\Leftrightarrow \alpha+1 < \sigma < (\alpha+\beta+2)/2)$. *Suppose furthermore that either* (4) *or* (5) *holds. Then, the same conclusion as in Theorem 2 holds.*

**Remark 1.** (i) In Theorem 1 the differentiability of $p$ is actually unnecessary. Similarly, in Theorems 2 and 3, the differentiability of $p$ is unnecessary when (4) is assumed.

(ii) When $\alpha = 1$ and $\varepsilon(t) \equiv 0$, Theorems 1, 2 and 3 have been obtained by [1] and [3, Corollaries 20.2 and 20.3].

We note that existence results of weakly increasing positive solutions to (1) and (E) are known for the case $\alpha > \beta$. In fact, equation (E) has a weakly increasing positive solution if and only if $\beta + 1 < \sigma \le \alpha + 1$; see Remark 2 in Section 3. In contrast, it seems that there are not such useful results for the case $\alpha < \beta$. But we can show many concrete examples of those equations that have weakly increasing positive solutions.

The paper is organized as follows. In Section 2 we give preparatory lemmas employed later. In Section 3 we consider equation (E), as well as (1), under the sub-homogeneity condition $\alpha > \beta$. When $q(t)$ is restricted to be functions such that $0 < \liminf_{t\to\infty} q(t)/t^{-\sigma} \le \limsup_{t\to\infty} q(t)/t^{-\sigma} < \infty$ for some $\sigma \in R$, we can obtain a result which may be called as asymptotic equivalence theorem for equation (1); see Corollary 8. Theorem 1 is a direct consequence of this corollary. In Section 4 we consider only equation (E) under the super-homogeneity condition $\alpha < \beta$. The proof of Theorems 2 and 3 will be given there. Other related results are found in [2,4,5,6].

## 2 Preparatory lemmas

**Lemma 4** *Let $w \in C^1[t_0,\infty), w'(t) = O(1)$ as $t \to \infty$, and $w \in L^\lambda[t_0,\infty)$ for some $\lambda > 0$. Then, $\lim_{t\to\infty} w(t) = 0$.*

**Proof.** We have

$$\begin{aligned} |w(t)|^\lambda w(t) &= |w(t_0)|^\lambda w(t_0) + \int_{t_0}^t (|w(s)|^\lambda w(s))' ds \\ &= |w(t_0)|^\lambda w(t_0) + (\lambda+1)\int_{t_0}^t |w(s)|^\lambda w'(s) ds. \end{aligned}$$

By our assumptions the last integral converges as $t \to \infty$. Hence $\lim_{t\to\infty} |w(t)|^\lambda w(t) \in R$ exists. Since $w \in L^\lambda[t_0,\infty)$, the limit must be 0. The proof is completed.

**Lemma 5** *Let $u \in C^1[t_0,\infty)$ and $u'(t) \downarrow 0$ as $t \to \infty$. Then, $tu'(t) \le u(t)$ for sufficiently large $t$, and the function $u(t)/t$ is decreasing near $\infty$.*

**Proof.** Since

$$u(t) = u(t_0) + \int_{t_0}^t u'(s)ds \ge u(t_0) + u'(t)(t - t_0),$$

we have

$$tu'(t) - u(t) \le t_0 u'(t) - u(t_0).$$

Noting the assumption $u'(\infty) = 0$ we find that $tu'(t) - u(t) < 0$ near $\infty$. Since $(u(t)/t)' = (tu'(t) - u(t))/t^2$, the proof is completed.

## 3 Sub-homogeneous case: $\alpha > \beta$

Throughtout the section we assume that $\alpha > \beta$. As a first step we give the growth estimates for weakly increasing positive solutions of (1):

**Lemma 6** *Let $u$ be a weakly increasing positive solution of (1). Then the following estimates hold near $\infty$:*

$$\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left\{\int_{t_1}^t \left(\int_s^\infty q(r)dr\right)^{\frac{1}{\alpha}} ds\right\}^{\frac{\alpha}{\alpha-\beta}} \le u(t)$$

$$\le \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left\{\int_{t_1}^t s^{-\frac{\beta}{\alpha}} \left(\int_s^\infty r^\beta q(r)dr\right)^{\frac{1}{\alpha}} ds\right\}^{\frac{\alpha}{\alpha-\beta}} \tag{6}$$

*where $t_1$ is a sufficiently large number.*

Note that $\int^{\infty} s^{\beta} q(s)ds < \infty$ if equation (1) has a weakly increasing positive solution; see (ii) of Remark 2 in the sequel.

**Proof of Lemma 6.** We may assume that $u, u' > 0$ for $t \geq t_1$. Since $u$ satisfies for large $t$

$$u'(t)^{\alpha} = \int_t^{\infty} q(s)u(s)^{\beta} ds, \tag{7}$$

and $u$ is increasing, we have

$$u'(t)^{\alpha} \geq u(t)^{\beta} \int_t^{\infty} q(s)ds,$$

that is

$$u'(t)u(t)^{-\frac{\beta}{\alpha}} \geq \left( \int_t^{\infty} q(s)ds \right)^{\frac{1}{\alpha}}. \tag{8}$$

An integration of this inequality on the interval $[t_1, t]$ will give

$$\frac{\alpha}{\alpha - \beta} \left\{ u(t)^{\frac{\alpha-\beta}{\alpha}} - u(t_1)^{\frac{\alpha-\beta}{\alpha}} \right\} \geq \int_{t_1}^{t} \left( \int_s^{\infty} q(r)dr \right)^{1/\alpha} ds,$$

which proves the first inequality of (6). On the other hand, by the decreasing nature of $u(t)/t$ shown in Lemma 5, we find from (7) that

$$u'(t)^{\alpha} \leq \left( \frac{u(t)}{t} \right)^{\beta} \int_t^{\infty} s^{\beta} q(s)ds.$$

Accordingly,

$$u'(t)u(t)^{-\frac{\beta}{\alpha}} \leq t^{-\frac{\beta}{\alpha}} \left( \int_t^{\infty} s^{\beta} q(s)ds \right)^{\frac{1}{\alpha}}.$$

As before we can get the second inequality in (6). The proof is completed.

To give the main result in the section let us consider the two equations of the same type:

$$(|u'|^{\alpha-1}u')' + q_1(t)|u|^{\beta-1}u = 0, \tag{$9_1$}$$

and

$$(|u'|^{\alpha-1}u')' + q_2(t)|u|^{\beta-1}u = 0. \tag{$9_2$}$$

Here, we assume that $0 < \alpha < \beta$ and $q_1, q_2 \in C([t_0, \infty); (0, \infty))$.

**Theorem 7** *Suppose that*

$$q_1(t) \sim q_2(t) \quad as \quad t \to \infty \tag{10}$$

*and*

$$C \int_{t_0}^{t} s^{-\frac{\beta}{\alpha}} \left( \int_s^{\infty} r^{\beta} q_1(r)dr \right)^{\frac{1}{\alpha}} ds \leq \int_{t_0}^{t} \left( \int_s^{\infty} q_1(r)dr \right)^{\frac{1}{\alpha}} ds \tag{11}$$

*hold for some constant $C > 0$. If $u_1$ and $u_2$ are weakly increasing positive solutions of equations $(9_1)$ and $(9_2)$, respectively, then $u_1(t) \sim u_2(t)$ as $t \to \infty$.*

**Corollary 8** *Suppose that* $q_1$ *and* $q_2$ *satisfy* (10) *and* $0 < \liminf_{t\to\infty} q_1(t)/t^{-\sigma} \leq \limsup_{t\to\infty} q_1(t)/t^{-\sigma} < \infty$ *for some* $\sigma \in (\beta+1, \alpha+1]$*. If* $u_1$ *and* $u_2$ *are weakly increasing positive solutions of equations* $(9_1)$ *and* $(9_2)$*, respectively, then* $u_1(t) \sim u_2(t)$ *as* $t \to \infty$*.*

Theorem 1 is an immediate consequence of Corollary 8. Indeed, to see (ii) of Theorem 1, it suffices to notice the fact that the equation

$$(|u'|^{\alpha-1}u')' + t^{-\alpha-1}\left(1 - \frac{\beta}{(\alpha-\beta)\log t}\right)|u|^{\beta-1}u = 0$$

has a weakly increasing positive solution given explicitly by

$$\alpha^{-\frac{1}{\alpha-\beta}}\left(\frac{\alpha}{\alpha-\beta}\right)^{-\frac{\alpha}{\alpha-\beta}}(\log t)^{\frac{\alpha}{\alpha-\beta}}.$$

**Proof of Theorem 7.** Put $z(t) = u_1(t)/u_2(t)$, $t \geq t_0$, $t_0$ being sufficiently large. Then, $z$ satisfies the equation

$$z'' + \frac{2u_2'(t)}{u_2(t)}z' + \frac{u_2(t)^{\beta-1}}{\alpha}\left[(u_2(t)z' + u_2'(t)z)^{1-\alpha}q_1(t)z^\beta - q_2(t)u_2'(t)^{1-\alpha}z\right] = 0.$$

If $z'(T) = 0$ for some $T$, then

$$z''(T) = \alpha^{-1}p_1(T)u_2(T)^{\beta-1}u_2'(T)^{1-\alpha}z(T)\left(\frac{q_2(T)}{q_1(T)} - z(T)^{\beta-\alpha}\right).$$

Thus, if $z' = 0$ in the region $z > (q_1(t)/q_2(t))^{1/(\alpha-\beta)}$, then $z$ attains a local minimum here; while if $z' = 0$ in the region $0 < z < (q_1(t)/q_2(t))^{1/(\alpha-\beta)}$, then $z$ attains a local maximum here. Note that by our assumption $\lim_{t\to\infty}(q_1(t)/q_2(t))^{1/(\alpha-\beta)} = 1$. These simple observations are used below.

Since Lemma 6 and conditions (10) and (11) imply that $z(t)$ is bounded and bounded from 0, we can put $0 < \ell = \liminf_{t\to\infty} z(t) \leq \limsup_{t\to\infty} z(t) = L < \infty$. We claim that $\ell = L$; that is $\lim_{t\to\infty} z(t) \in (0,\infty)$ exists. Suppose the contrary that $\ell \neq L$. We treat the following four cases separatively: (a)$L \geq 1 > \ell$; (b)$L > 1 \geq \ell$; (c)$L > \ell \geq 1$; (d)$1 \geq L > \ell$.

Suppose that case (a) occurs. We can find two sequences $\{T_n\}$ and $\{t_n\}$ satisfying

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} t_n = \infty \tag{12}$$

and

$$\lim_{n\to\infty} z(T_n) = L, \quad \lim_{n\to\infty} z(t_n) = \ell, \quad \text{and} \quad t_n < T_n < t_{n+1} \quad \text{for} \quad n = 1, 2, \cdots. \tag{13}$$

Since $\lim_{t\to\infty}(q_1(t)/q_2(t))^{1/(\alpha-\beta)} = 1$, we may assume that $z(t_n) < (q_1(t_n)/q_2(t_n))^{1/(\alpha-\beta)}$. For sufficiently largr $n \in N$ the minimum of $z(t)$ on the interval $[T_{n-1}, T_n]$ must be attained at an interior point, say $t_* \in (T_{n-1}, T_n)$. Obviously, $z'(t_*) = 0$ and $z''(t_*) \geq 0$. However, since $z(t_*) \leq z(t_n)$ for sufficiently large $n$, we get a contradiction to the above observation. Hence case (a) does not occur.

Next suppose that case (c) occurs. As in case (a) we can find two sequences $\{T_n\}$ and $\{t_n\}$ satisfying (12) and (13). For sufficiently lage $n \in N$ the maximum of $z(t)$ on the interval $[t_n, t_{n+1}]$ must be attained at an interior point, say $t^* \in (t_n, t_{n+1})$. Obviously, $z'(t^*) = 0$ and $z''(t^*) \le 0$. Since $z(t^*) \ge z(T_n)$ and $z(T_n) > (q_1(T_n)/q_2(T_n))^{1/(\alpha-\beta)}$ for sufficiently large $n$, we get a contradiction as before. Hence case (c) does not occur.

We can show similarily that the other cases can not occur. Therefore $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} u_2(t)/u_1(t) = m \in (0, \infty)$ exists. Finally, by the L'Hospital's rule we have

$$m = \lim_{t\to\infty} \frac{u_1(t)}{u_2(t)} = \left( \lim_{t\to\infty} \frac{u_1'(t)^\alpha}{u_2'(t)^\alpha} \right)^{1/\alpha} = \left( \lim_{t\to\infty} \frac{[u_1'(t)^\alpha]'}{[u_2'(t)^\alpha]'} \right)^{1/\alpha} = \left( \lim_{t\to\infty} \frac{-q_1(t)u_1(t)^\beta}{-q_2(t)u_2(t)^\beta} \right)^{1/\alpha} = m^{\beta/\alpha};$$

that is $m = m^{\beta/\alpha}$. Since $\alpha > \beta$, we have $m = 1$. This completes the proof.

**Remark 2.** Concerning the existence properties of weakly increasing positive solutions, we know the following results:

(i) If

$$\int^\infty t^\beta q(t)dt < \infty; \quad \text{and} \quad \int^\infty \left( \int_t^\infty q(s)ds \right)^{1/\alpha} dt = \infty, \tag{14}$$

Then, equation (1) has weakly increasing positive solutions [2, Example 1].

(ii) Conversely, if equation (1) has a weakly increasing positive solution, then we can show that

$$\int^\infty t^\beta q(t)dt < \infty; \quad \text{and} \quad \int^\infty t^{-\frac{\beta}{\alpha}} \left( \int_t^\infty r^\beta q(s)ds \right)^{\frac{1}{\alpha}} ds = \infty. \tag{15}$$

In fact, the first condition in (15) follows from [2, Example 2]; while the second one is an immediate consequence of the estimates in Lemma 6. In paricular, when $\alpha = 1$, we find that conditions (14) and (15) are the same; that is, equation (1) (with $\alpha = 1$) has a weakly increasing positive solution if and only if (14) (with $\alpha = 1$) holds.

# 4 Super-homogeneous case: $\alpha < \beta$

Throughout this section we assume that $\alpha < \beta$. In this case the situation seems to be more complicated than in the previous case. The main purpose of the section is to give the proofs of Theorems 2 and 3. To this end we need several lemmas.

**Lemma 9** *Let* $0 < \liminf_{t\to\infty} q(t)/t^{-\sigma} \le \limsup_{t\to\infty} q(t)/t^{-\sigma} < \infty$ *for some* $\sigma \in (\alpha + 1, \beta + 1)$. *Then every weakly increasing positive solution* $u$ *of* (E) *satisfies* $u(t) = O(u_0(t))$ *and* $u'(t) = O(u_0'(t))$ *as* $t \to \infty$, *where* $u_0$ *is the exact solution of* (E$_0$) *given by* (2) *and* (3).

**Proof.** As in the proof of Lemma 6 we obtain (8). An integration of (8) on the interval $[t, \infty)$ will give

$$u(t) \le C_1 \left\{ \int_t^\infty \left( \int_s^\infty q(r)dr \right)^{1/\alpha} ds \right\}^{-\alpha/(\beta-\alpha)} \equiv C_2 u_0(t),$$

where $C_1$ and $C_2$ are positive constant. Furthermore, by (7) we find that

$$u'(t) = \left( \int_t^\infty q(s)u(s)^\beta ds \right)^{1/\alpha} \le C_3 \int_t^\infty s^{-\sigma+k\beta} ds = C_4 t^{k-1} = O(u_0'(t)) \quad \text{as} \quad t \to \infty,$$

where $C_3$ and $C_4$ are positive constants. This completes the proof.

**Lemma 10** *Let $\sigma \in (\alpha+1, \beta+1)$, and $u$ a weakly increasing positive solution of equation* (E). *Put $s = \log u_0(t)$ and $v = u/u_0$. Then*
(i) $v, \dot{v} = O(1)$ *as $s \to \infty$, and $v + \dot{v} > 0$ near $\infty$, where $\cdot = d/ds$;*
(ii) $v(s)$ *satisfies near $\infty$ the equation*

$$\ddot{v} + a\dot{v} - bv + b(\dot{v} + v)^{1-\alpha} v^\beta + b\delta(s)(\dot{v} + v)^{1-\alpha} v^\beta = 0, \tag{16}$$

*where*

$$a = 2 - \frac{1}{k} \ne 0, \quad b = \frac{1-k}{k} > 0, \quad \text{and} \quad \delta(s) = \varepsilon(t).$$

**Proof.** We will prove only (i), because (ii) can be proved by direct computations. We assume that $u, u' > 0$. Since $u = u_0 v$, the boundedness of $v$ follows from Lemma 9. Noting $du/dt = \hat{C} k t^{k-1}(v + \dot{v})$, we have $v + \dot{v} > 0$. On the other hand, since $dt/ds = t/k$, we have

$$|\dot{v}| = \left| \frac{d}{dt} \left( \frac{u}{u_0} \right) \frac{dt}{ds} \right| = \left| \frac{u' u_0 - u_0' u}{u_0^2} \right| \frac{t}{k} \le C \frac{t^{k-1} t^k t}{t^{2k}} = O(1) \quad \text{as} \quad s \to \infty.$$

This completes the proof.

**Lemma 11** *Let the assumption either of Theorem 2 or Theorem 3 holds, and $v$ be as in Lemma 10. Then $\dot{v} \in L^2[s_0, \infty)$ for sufficiently large $s_0$.*

**Proof.** We note that conditions (4) and (5), respectively, are equivalent to

$$\int^\infty \delta(s)^2 ds < \infty \tag{17}$$

and

$$\int^\infty |\dot{\delta}(s)| ds < \infty. \tag{18}$$

We firstly consider the case where assumptions of Theorem 2 hold. In this case, the constant $a$ appearing in (16) is positive.

We multiply the both sides of (16) by $\dot{v}$. Since $\alpha \le 1$, we have $(1 + \delta(s))(\dot{v} + v)^{1-\alpha}\dot{v} \ge (1 + \delta(s))v^{1-\alpha}\dot{v}$; and so we obtain

$$\ddot{v}\dot{v} + a\dot{v}^2 - bv\dot{v} + bv^{1-\alpha+\beta}\dot{v} + b\delta(s)v^{1-\alpha+\beta}\dot{v} \le 0. \tag{19}$$

An integration gives

$$\frac{\dot{v}^2}{2} + a\int_{s_0}^s \dot{v}^2 dr - \frac{b}{2}v^2 + \frac{bv^{2-\alpha+\beta}}{2-\alpha+\beta} + b\int_{s_0}^s \delta(r)v^{1-\alpha+\beta}\dot{v} dr \le \text{const}; \tag{20}$$

that is

$$a\int_{s_0}^{s}\dot v^2dr+b\int_{s_0}^{s}\delta(r)v^{1-\alpha+\beta}\dot v dr\le O(1)\quad\text{as}\quad s\to\infty.$$

Here we have employed (i) of Lemma 10. Let the integral condition (4) hold; that is, let (17) hold. By the Schwarz inequality we have

$$a\int_{s_0}^{s}\dot v^2dr-C_1\left(\int_{s_0}^{\infty}\delta(r)^2dr\right)^{1/2}\left(\int_{s_0}^{s}\dot v^2dr\right)^{1/2}\le O(1)$$

for some constant $C_1>0$. Therefore $\dot v\in L^2[s_0,\infty)$. Next let (5) hold. Using integral by parts, we obtain from (20)

$$\frac{\dot v^2}{2}+a\int_{s_0}^{s}\dot v^2dr-\frac{b}{2}v^2+\frac{b[1+\delta(r)]v^{2-\alpha+\beta}}{2-\alpha+\beta}-\frac{b}{2-\alpha+\beta}\int_{s_0}^{s}\dot\delta(r)v^{2-\alpha+\beta}dr\le\text{const.}$$

Noting (i) of Lemma 10, we find that $\dot v\in L^2[s_0,\infty)$.

Secondly we consider the case where assumptions of Theorem 3 hold. As above, we multiply the both sides of (16) by $\dot v$. Since $\alpha\ge1$, we have $(1+\delta(s))(v+\dot v)^{1-\alpha}\dot v\le(1+\delta(s))v^{1-\alpha}\dot v$, and so we obtain

$$|a|\dot v^2\le\dot v\ddot v+bv\dot v+b(1+\delta(s))v^{1-\alpha+\beta}\dot v.$$

An integration on the interval $[s_0,s]$ gives

$$|a|\int_{s_0}^{s}\dot v^2dr\le\frac{\dot v^2}{2}+\frac{bv^2}{2}+\frac{bv^{2-\alpha+\beta}}{2-\alpha+\beta}+\int_{s_0}^{s}\delta(r)v^{1-\alpha+\beta}\dot v dr+\text{const.}$$

As before, we will obtain $\dot v\in L^2[s_0,\infty)$. This completes the proof.

**Proof of Theorem 2.** To this end it suffices to show that $\lim_{s\to\infty}v(s)=1$, where $v(s)$ is the function introduced in Lemma 10. The proof is divided into three steps.

*Step 1.* We claim that $\liminf_{s\to\infty}v(s)>0$; namely $\liminf_{t\to\infty}u(t)/u_0(t)>0$. The proof is done by contradiction. Suppose to the contrary that $\liminf_{s\to\infty}v(s)=0$. Firstly, we suppose that $v(s)$ decrease to 0 as $s\to\infty$. This means that $u(t)/u_0(t)$ decreases to 0 as $t\to\infty$. Accordingly we have

$$u'(t)^\alpha=\int_t^\infty p(r)u(r)^\beta dr=\int_t^\infty p(r)u_0(r)^\beta\left(\frac{u(r)}{u_0(r)}\right)^\beta dr$$

$$\le\left(\frac{u(t)}{u_0(t)}\right)^\beta\int_t^\infty p(r)u_0(r)^\beta dr=C_1t^{1-\sigma}u(t)^\beta,$$

where $C_1>0$ is a constant. Consequently we obtain the differential inequality $u'\le C_2t^{(1-\sigma)/\alpha}u^{\beta/\alpha}$for some constant $C_2>0$ near $\infty$. But this differential inequality implies that $u(t)/u_0(t)\equiv v(s)\ge C_3>0$ for some constant $C_3>0$. This is an obvious contradiction. Next suppose that $\liminf_{s\to\infty}v(s)=0$ and $\dot v(s)$ changes the sign in any neighborhood of $\infty$. We notice that, if $\dot v=0$ in the region $0<v<[1+\delta(s)]^{-1/(\beta-\alpha)}$, then

$\ddot{v} > 0$; while if $\dot{v} = 0$ in the region $v > [1 + \delta(s)]^{-1/(\beta-\alpha)}$, then $\ddot{v} < 0$. Therefore in this case the curve $v = v(s)$ must cross the curve $v = [1 + \delta(s)]^{-1/(\beta-\alpha)}$ infinitely many times as $s \to \infty$. Therefore, we can find out two sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfying

$$\xi_n < \eta_n < \xi_{n+1}, \quad n = 1, 2, \cdots ; \quad \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n = \infty$$

and

$$v(\eta_n) \to 0 \quad \text{as} \quad n \to \infty, \quad v(\xi_n) = [1 + \delta(\xi_n)]^{\frac{-1}{\beta-\alpha}} \to 1 \quad \text{as} \quad n \to \infty.$$

An integration of (19) on $[\xi_n, \eta_n]$ yields

$$\frac{1}{2}\{\dot{v}(\eta_n)^2 - \dot{v}(\xi_n)^2\} + a\int_{\xi_n}^{\eta_n} \dot{v}^2 dr - \frac{b}{2}\{v(\eta_n)^2 - v(\xi_n)^2\}$$

$$+\frac{b}{2-\alpha+\beta}\{v(\eta_n)^{2-\alpha+\beta} - v(\xi_n)^{2-\alpha+\beta}\} + b\int_{\xi_n}^{\eta_n} \delta(r)v^{1-\alpha+\beta}\dot{v}dr \leq 0. \tag{21}$$

From equation (16) and (i) of Lemma 10 we know that $\ddot{v} = O(1)$ as $s \to \infty$. This fact and the fact that $\dot{v} \in L^2[s_0, \infty)$ imply that $\lim_{s\to\infty} \dot{v}(s) = 0$ by Lemma 4. Accordingly (21) is equivalent to

$$o(1) + o(1) - \frac{b}{2}(o(1) - 1) + \frac{b}{2-\alpha+\beta}(o(1) - 1) + o(1) \leq 0 \quad \text{as} \quad s \to \infty.$$

Letting $n \to \infty$, we get $b/2 - b/(2 - \alpha + \beta) \leq 0$ a contradiction to the assumption $\beta > \alpha$. Therefore, $\liminf_{s\to\infty} v(s) > 0$.

*Step 2.* We claim that there is a limit $\lim_{s\to\infty} v(s) \in (0, \infty)$. To see this, we integrate (16) multiplied by $\dot{v}$:

$$\frac{\dot{v}^2}{2} + a\int_{s_0}^{s} \dot{v}^2 dr - \frac{b}{2}v^2 + b\int_{s_0}^{s} (\dot{v} + v)^{1-\alpha}v^{\beta}\dot{v}dr$$

$$+b\int_{s_0}^{s} \delta(r)(\dot{v} + v)^{1-\alpha}v^{\beta}\dot{v}dr = \text{const.} \tag{22}$$

Suppose that condition (4); namely (17) holds. Since $\dot{v} \in L^2[s_0, \infty)$, the first, and the third integrals in the left hand side of (22) converge as $s \to \infty$. The mean value theorem shows that

$$(v + \dot{v})^{1-\alpha} = \left(1 + \frac{\dot{v}}{v}\right)^{1-\alpha} v^{1-\alpha} = v^{1-\alpha} + (1 - \alpha)\,(v + \theta(r)\dot{v})^{-\alpha}\,\dot{v}, \tag{23}$$

where $\theta(r)$ is a quantity satisfying $0 < \theta(r) < 1$. Therefore,

$$\int_{s_0}^{s} (\dot{v} + v)^{1-\alpha}v^{\beta}\dot{v}dr = \int_{s_0}^{s} \{v^{1-\alpha+\beta}\dot{v} + (1 - \alpha)(v + \theta(r)\dot{v})^{-\alpha}v^{\beta}\dot{v}^2\}dr$$

$$= \frac{v(s)^{2+\beta-\alpha}}{2+\beta-\alpha} + \int_{s_0}^{s} O(1)\dot{v}^2 dr + \text{const.}$$

So we find that the function $-v^2/2 + v^{2+\beta-\alpha}/(2+\beta-\alpha)$ has a finite limit. This fact shows that $\lim_{s\to\infty} v(s) = m \in (0,\infty)$ exists. Suppose that (5); namely (18) holds. We have by (23)

$$\int_{s_0}^{s} \delta(r)(\dot{v}+v)^{1-\alpha}v^{\beta}\dot{v}dr = \int_{s_0}^{s} \{\delta(r)v^{1-\alpha+\beta}\dot{v} + \delta(r)(1-\alpha)(v+\theta(r)\dot{v})^{-\alpha}v^{\beta}\dot{v}^2\}dr$$

$$= \frac{\delta(s)v^{2+\beta-\alpha}}{2+\beta-\alpha} - \frac{1}{2+\beta-\alpha}\int_{s_0}^{s} \dot{\delta}(r)v^{2+\beta-\alpha}dr + \text{const} + \int_{s_0}^{s} O(1)\dot{v}^2 dr$$

as $s \to \infty$. Hence, as before we know that the function $-v^2/2 + v^{2+\beta-\alpha}/(2+\beta-\alpha)$ has a finite limit. Therefore $m = \lim_{s\to\infty} v(s) \in (0,\infty)$ exists.

*Step 3.* Finally, we let $s \to \infty$ in equation (16). Then, we have $\lim_{s\to\infty} \ddot{v}(s) = b(m - m^{1+\beta-\alpha})$. Since $\dot{v} = o(1)$, we must have $\lim_{s\to\infty} \ddot{v}(s) = 0$, implying $m = 1$. The proof of Theorem 2 is completed.

**Proof of Theorem 3.** As in the proof of Theorem 2, firstly we show that $\liminf_{s\to\infty} v(s) > 0$. The proof is done by contradiction. Suppose to the contrary that $\liminf_{s\to\infty} v(s) = 0$. We may assume that $\dot{v}$ changes the sign in any neighborhood of $\infty$. Since $v(s)$ takes local maxima in the region $v \geq (1+\delta(s))^{-1/(\beta-\alpha)}$, there are the following sequences $\{\underline{s}_n\}$ and $\{\overline{s}_n\}$ satisfying

$$\underline{s}_n < \overline{s}_n < \underline{s}_{n+1}, \quad \lim_{n\to\infty} \underline{s}_n = \lim_{n\to\infty} \overline{s}_n = \infty$$

and

$$\dot{v}(\underline{s}_n) = \dot{v}(\overline{s}_n) = 0, \quad \lim_{n\to\infty} v(\underline{s}_n) = 0, \quad v(\overline{s}_n) \geq (1+\delta(\overline{s}_n))^{-1/(\beta-\alpha)}.$$

Now, we decompose $\alpha$ in the form $\alpha = m - \rho$, where $m \in N$ and $\rho > 0$. Although there are infinitely many such choices of decomposition for $\alpha$, we fix one choice for a moment. We rewrite equation (16) as

$$\ddot{v} - |a|\dot{v} - bv + b(\dot{v}+v)^{-m+1+\rho}v^{\beta} + b\delta(s)(\dot{v}+v)^{-m+1+\rho}v^{\beta} = 0.$$

We multiply the both sides by $(v+\dot{v})^m\dot{v}$ and then integrate the resulting equation on the interval $[\underline{s}_n, \overline{s}_n]$ to obtain

$$\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}(v+\dot{v})^m dr - |a|\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^m\dot{v}^2 dr - b\int_{\underline{s}_n}^{\overline{s}_n} v\dot{v}(v+\dot{v})^m dr$$

$$+b\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^{1+\rho}\dot{v}v^{\beta}dr + b\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)(v+\dot{v})^{1+\rho}\dot{v}v^{\beta}dr = 0. \tag{24}$$

The binomial expansion implies that

$$\sum_{k=0}^{m} c_k \int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1}v^{m-k}dr - |a|\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^m\dot{v}^2 dr - b\sum_{k=0}^{m} c_k \int_{\underline{s}_n}^{\overline{s}_n} v^{m-k+1}\dot{v}^{k+1}dr$$

$$+b\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^{1+\rho}\dot{v}v^{\beta}dr + b\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)(v+\dot{v})^{1+\rho}\dot{v}v^{\beta}dr = 0,$$

where $c_k = \binom{m}{k}$ are the binomial coefficients. Now, we evaluate each term in the left hand side. For $k \in \{0, 1, ..., m-1\}$ we obtain

$$\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1} v^{m-k} dr = \int_{\underline{s}_n}^{\overline{s}_n} \frac{d}{dr}\left(\frac{\dot{v}^{k+2}}{k+2}\right) v^{m-k} dr$$

$$= -\frac{m-k}{k+2} \int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^{k+3} v^{m-k-1} dr = o(1) \quad \text{as} \quad n \to \infty.$$

For $k = m$ obviously we have $\int_{\underline{s}_n}^{\overline{s}_n} \ddot{v}\dot{v}^{k+1} dr = 0$. Hence the first term of the left hand side of (24) tends to 0 as $n \to \infty$. The second term is dominated by Const$\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr$, and hence it tends to zero as $n \to \infty$. Next, we compute the third term. For $k \in \{1, 2, ..., m\}$ we have $|\int_{\underline{s}_n}^{\overline{s}_n} v^{m-k+1}\dot{v}^{k+1} dr| \le$ const $\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr$. For $k = 0$ we have

$$\int_{\underline{s}_n}^{\overline{s}_n} v^{m+1}\dot{v} dr = \frac{1}{m+2}\left(v(\overline{s}_n)^{m+2} - v(\underline{s}_n)^{m+2}\right) = \frac{v(\overline{s}_n)^{m+2}}{m+2} + o(1) \quad \text{as} \quad n \to \infty.$$

Therefore the third term is equal to

$$o(1) - \frac{bv(\overline{s}_n)^{m+2}}{m+2} \quad \text{as} \quad n \to \infty.$$

To evaluate the fourth term we employ the mean value theorem to obtain

$$(v + \dot{v})^{1+\rho} = v^{1+\rho} + (1+\rho)\,(v + \theta(r)\dot{v})^{\rho}\,\dot{v},$$

where $\theta(r)$ is a quantity between 0 and 1. Hence we can compute

$$\int_{\underline{s}_n}^{\overline{s}_n} (v+\dot{v})^{1+\rho}\dot{v}v^{\beta} dr = \int_{\underline{s}_n}^{\overline{s}_n} v^{1+\rho+\beta}\dot{v} dr + (1+\rho)\int_{\underline{s}_n}^{\overline{s}_n} (v + \theta(r)\dot{v})^{\rho}\,\dot{v}^2 v^{\beta} dr$$

$$= \frac{v(\overline{s}_n)^{2+\rho+\beta} - v(\underline{s}_n)^{2+\rho+\beta}}{2+\rho+\beta} + (1+\rho)\int_{\underline{s}_n}^{\overline{s}_n} O(1)\dot{v}^2 dr = \frac{v(\overline{s}_n)^{2+\rho+\beta}}{2+\rho+\beta} + o(1) \quad \text{as} \quad n \to \infty.$$

Finally by Schwarz's inequality we find that the last term is dominated by the quantity

$$\text{const}\left(\int_{\underline{s}_n}^{\overline{s}_n} \delta(r)^2 dr\right)^{1/2}\left(\int_{\underline{s}_n}^{\overline{s}_n} \dot{v}^2 dr\right)^{1/2} = o(1) \quad \text{as} \quad n \to \infty.$$

Consequently, from (24) we obtain the formula

$$o(1) - \frac{b}{m+2} v(\overline{s}_n)^{m+2} + \frac{b}{2+\rho+\beta} v(\overline{s}_n)^{2+\rho+\beta} + o(1) = 0 \quad \text{as} \quad n \to \infty.$$

This implies that $\lim_{n\to\infty} v(\overline{s}_n) = [(m+2+\beta-\alpha)/(m+2)]^{1/\beta}$. Since $m$ can be moved arbitrarily, this is an obvious contradiction. Therefore $\liminf_{s\to\infty} v > 0$.

We are now in a position to show $\lim_{s\to\infty} v(s) = 1$. Since $\liminf_{s\to\infty} v(s) > 0$, we find that $\liminf_{t\to\infty} u(t)/u_0(t) > 0$. Integrating equation (7) (with $q$ replaced by $p$), we further find that $\liminf_{t\to\infty} u'(t)/u_0'(t) > 0$. Since $v + \dot{v} = u'(t)/u_0'(t)$, we obtain $\liminf_{s\to\infty}(v + \dot{v}) > 0$. Recalling equation (16), we find that $\ddot{v}(s) = O(1)$ as $s \to \infty$. Since we have already known that $\dot{v} \in L^2[s_0, \infty)$, Lemma 4 shows that $\lim_{s\to\infty} \dot{v} = 0$. Concequently, as in the proof of Theorem 2 we can prove that $\lim_{s\to\infty} v(s) = 1$. This completes the proof of Theorem 3.

## References


[1] R. Bellman, Stability theory of differential equations, MaGraw-Hill, New York, 1953. (Reprint: Dover Publications, Inc., 1969)

[2] Á. Elbert and T. Kusano, Oscillation and non-oscillation theorems for a class of second order quasilinear differential equations, Acta Math. Hung. 56 (1990), 325-336.

[3] I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations, Kluwer Academic Publisher, Dordrecht, 1992.

[4] M. Kitano and T. Kusano, On a class of second order quasilinear ordinary differential equations, Hiroshima Math. J. 25 (1995), 321-355.

[5] R. A. Moore and Z. Nehari, Nonoscillation theorems for a class of nonlinear differential equations, Trans. Amer. Math. Soc. 93 (1959), 30-52.

[6] K. Nishihara, Asymptotic behaviors of solutions of second order differential equations, J. Math. Anal. Appl. 189 (1995), 424-441.



Ken-ichi Kamo:
Division of Mathematics, School of Medicine, Liberal Arts and Sciences
Sapporo Medical University
S1W17, Chuoku, Sapporo 060-8543, Japan
kamo@sapmed.ac.jp

Hiroyuki Usami:
Department of Mathematics, Graduate School of Science,
Hiroshima University,
Higashi-Hiroshima, 739-8521, Japan
usami@mis.hiroshima-u.ac.jp