On the variety of 3-dimensional Lie algebras

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Abstract

It is known that a 3-dimensional Lie algebra is unimodular or solvable as a result of the classification. We give a simple proof of this fact, based on a fundamental identity for 3-dimensional Lie algebras, which was first appeared in [21]. We also give a representation theoretic meaning of the invariant of 3-dimensional Lie algebras introduced in [15], [22], by calculating the $GL(V)$-irreducible decomposition of polynomials on the space $\wedge^2 V^* \otimes V$ up to degree 3. Typical four covariants naturally appear in this decomposition, and we show that the isomorphism classes of 3-dimensional Lie algebras are completely determined by the $GL(V)$-invariant concepts in $\wedge^2 V^* \otimes V$ defined by these four covariants. We also exhibit an explicit algorithm to distinguish them.

Introduction

In this survey paper, we consider a variety consisting of 3-dimensional Lie algebras from the representation theoretic viewpoint, and give some perspective to the understanding of the set of 3-dimensional Lie algebras.

As is well known, the set of Lie algebra structure on a fixed vector space V constitutes a $GL(V)$-invariant algebraic set in $\wedge^2 V^* \otimes V$, whose defining equations are the quadratic polynomials called the Jacobi identity: $\mathfrak{S}\ [[X,Y],Z] = 0$. (The symbol $\mathfrak{S}$ implies the cyclic sum.) In general, this algebraic set is not irreducible, and the number of irreducible components is known for low dimensional cases (cf. [7], [11], [15], [19]).

In the case of dim $V = 3$, it is decomposed into two irreducible components, one of which consists of unimodular Lie algebras, and the other of which consists of solvable Lie algebras. In this paper, we give a simple proof of this fact without using the classification of Lie algebras, but instead, by showing the fundamental identity

$$\mathfrak{S}\ (\mathrm{Tr\ ad}\ X) \cdot [Y,Z] = 0,$$

which is a peculiar phenomenon for 3-dimensional case.

The second purpose of this paper is to describe the set of 3-dimensional Lie algebras from $GL(V)$-invariant representation theoretic viewpoint. We first give a $GL(V)$-irreducible decomposition of the set of polynomials on $\wedge^2 V^* \otimes V$ up to degree 3, by using Littlewood-Richardson's rule. And next, we show that among the generators of these irreducible components, there exist four fundamental generators (= covariants), by which we can describe several concepts in a $GL(V)$-invariant way. For example, we can give a representation theoretic meaning of the invariant of 3-dimensional Lie algebras introduced in [15], [22]. And in addition, we can state an algorithm to distinguish the isomorphism classes of 3-dimensional Lie algebras, including the mutual relation of $GL(V)$-orbits and their degenerations, in terms of these four covariants.

Of course, we already know the classification of 3-dimensional Lie algebras, and their individual structures are well known. But by this $GL(V)$-invariant viewpoint, we can obtain a systematic understanding of the set of 3-dimensional Lie algebras, which cannot be read from a mere classification, constituting a dense forest of mutually unrelated normal forms. The explicit description of these results are summarized at the end of § 4. We remark that our viewpoint in this paper is quite effective in considering several $GL(V)$-invariant problems in geometry. For other explicit examples, see [4], [5], [6].

Throughout this paper, we assume that Lie algebras and vector spaces are defined over the field of complex numbers $\mathbf{C}$, unless otherwise stated.

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§ 1. **Fundamental identity.**

We fix a basis $X_1, \cdots, X_n$ of an n-dimensional vector space V, and consider a skew symmetric bilinear map $[\ ,\ ] : V \times V \to V$ defined by $[X_i,\ X_j] = \sum c_{ij}^k X_k$ $(c_{ij}^k = -c_{ji}^k)$. Then, the bilinear map $[\ ,\ ]$ defines a Lie algebra structure on V if and only if it satisfies the Jacobi identity

$$\sum_l (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0.$$

This implies that the set of Lie algebra structures on V are defined by quadratic polynomial relations on c_{ij}^k.

In the case of dim $V = 3$, the Jacobi identity has another peculiar expression, from which the fundamental identity stated in Introduction follows.

Theorem 1. *The following identity holds for any* $[\ ,\ ] \in \wedge^2 V^* \otimes V$ *where* V *is a 3-dimensional vector space.*

$$\mathfrak{S}\ [[X_1, X_2], X_3] = \mathfrak{S}\ (\mathrm{Tr}\ \mathrm{ad} X_1) \cdot [X_2, X_3].$$

Proof. We may assume that X_1, X_2, X_3 are linearly independent because both expressions of the above equality reduce to zero in case X_1, X_2, X_3 are linearly dependent. We express $[X_i, X_j] = \sum c_{ij}^k X_k$ as above. Then, by direct calculations, we can easily show that both sides of the above equality are equal to

$$\begin{aligned} & (c_{12}^2 c_{23}^1 - c_{12}^1 c_{23}^2 + c_{13}^3 c_{23}^1 - c_{13}^1 c_{23}^3) X_1 + (c_{12}^1 c_{13}^2 - c_{12}^2 c_{13}^1 + c_{13}^3 c_{23}^2 - c_{13}^2 c_{23}^3) X_2 \\ + \ & (c_{12}^1 c_{13}^3 - c_{12}^3 c_{13}^1 + c_{12}^2 c_{23}^3 - c_{12}^3 c_{23}^2) X_3. \end{aligned}$$

But, we give here a representation theoretic proof, using the Schur functions. (For the definition and the fundamental properties of Schur functions, see [17].) We fix a volume form ω on V throughout. Then, we have a natural isomorphism $V \to \wedge^2 V^*$ defined by $X \to X \rfloor \omega$, and the bilinear map $[\ ,\ ] = \{c_{ij}^k\}$ may be considered as an element of $V \otimes V$ under this isomorphism.

In terms of the above basis of V, both sides of the equality in Theorem 1 are expressed as quadratic polynomials of c_{ij}^k, and they are considered as elements of $S^2(V \otimes V)^*$. The group $SL(V)$ which preserves the volume form ω naturally acts on $S^2(V \otimes V)^*$, and in the case $\dim V = 3$, from Littlewood-Richardson's rule, we have the following irreducible decomposition of $S^2(V \otimes V)^*$:

$$\begin{aligned} S^2(V \otimes V)^* &= S^2(V^* \otimes V^*) \\ &= S^2(S^2 V^* + \wedge^2 V^*) \\ &= S^2(S^2 V^*) + S^2 V^* \otimes \wedge^2 V^* + S^2(\wedge^2 V^*) \\ &= (S_4 + S_{22}) + (S_{31} + S_{211}) + S_{22} \\ &= S_4 + S_{31} + 2\, S_{22} + S_{211}. \end{aligned}$$

Here, S_λ implies the dual of the irreducible representation space of $SL(V)$ corresponding to the partition λ. (For example, the symbols S_1 and S_2 correspond to V^* and $S^2 V^*$, respectively. Strictly speaking, the dual space V^* must be represented as $S_{0 \cdots 0 -1}$. But we use this dual notations for simplicity. For details, see [17], [3].) Since the multiplicity of S_{211} in $S^2(V \otimes V)^*$ is one, and the space S_{211} is isomorphic to S_1 as an $SL(V)$-module, we have $\dim Hom_{SL(V)}(S^2(V \otimes V), V) = 1$. For $c = [\ ,\ ] \in \wedge^2 V^* \otimes V \cong V \otimes V$, we define two elements $\Phi(c)$, $\Psi(c) \in V$ by

$$\begin{aligned} \mathfrak{S}\ [[X_1, X_2], X_3] &= \omega(X_1, X_2, X_3)\Phi(c), \\ \mathfrak{S}\ (\mathrm{Tr\ ad}\ X_1) \cdot [X_2, X_3] &= \omega(X_1, X_2, X_3)\Psi(c). \end{aligned}$$

Then, we have Φ, $\Psi \in Hom_{SL(V)}(S^2(V \otimes V), V)$, and they must be proportional. The proportional constant does not depend on a specific $[\ ,\ ]$, and hence, by considering the bilinear map defined by

$$[X_1, X_2] = X_2,\ [X_2, X_3] = X_1,\ [X_3, X_1] = 0,$$

we can easily show that this constant is equal to 1. q.e.d.

By this theorem, we obtain the fundamental identity for 3-dimensional Lie algebras:

$$\mathfrak{S}\ (\mathrm{Tr\ ad}\ X_1) \cdot [X_2, X_3] = 0.$$

From this identity, we know that $\mathrm{Tr}\,\mathrm{ad}\,X = 0$ for any $X \in \mathfrak{g}$ if $\dim\,[\mathfrak{g},\mathfrak{g}] = 3$. Hence, we obtain the following result without the help of the classification of 3-dimensional Lie algebras. (In the following, we often express the space V as $\mathfrak{g}$ if a Lie algebra structure on V is explicitly or implicitly given.)

Corollary 2. *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra. Then $\mathfrak{g}$ satisfies* $\mathrm{Tr}\,\mathrm{ad}\,X = 0$ *for any $X \in \mathfrak{g}$, or* $\dim\,[\mathfrak{g},\mathfrak{g}] \leq 2$. *Namely, $\mathfrak{g}$ is unimodular or solvable in the case* $\dim \mathfrak{g} = 3$.

Remark. (1) The above theorem and corollary hold over any field k of characteristic $\neq 2$. As far as the author knows, the above fundamental identity was first appeared in the (unpublished) paper [21].

(2) In the case of $\dim V \geq 4$, in contrast with the case of $\dim V = 3$, the set of quadratic polynomials appearing in the Jacobi identity is not irreducible, and constitutes two $GL(V)$-irreducible components of $S^2(\wedge^2 V^* \otimes V)^*$, whose dimensions are $1/6 \cdot n(n-1)(n+1)(n-3)$ and $1/2 \cdot n(n-1)$. And hence, the argument in the proof of Theorem 1 does not hold for $\dim V \geq 4$. For example, we can easily check that $\mathfrak{S}$ $(\mathrm{Tr}\,\mathrm{ad}\,X_1) \cdot [X_2, X_3] \neq 0$ for the 4-dimensional Lie algebra defined by

$$[X_1, X_2] = X_2, \;\; [X_1, X_3] = X_3, \;\; [X_1, X_4] = 2\, X_4, \;\; [X_2, X_3] = X_4.$$

It seems that the fundamental identity of the above type directly related to the Jacobi identity does not exist in the case $\dim V \geq 4$.

Next, we consider the set of 3-dimensional Lie algebras in $\wedge^2 V^* \otimes V$. We already know the following result.

Theorem 3 (cf. [7], [11], [15], [19]). *The set of 3-dimensional Lie algebra structures on V constitutes two irreducible varieties in $\wedge^2 V^* \otimes V$, one of which is the closure of the set of simple Lie algebras and the other of which is the set of solvable Lie algebras.*

In the following, we denote these varieties by Σ_{simp} and Σ_{solv}, respectively. They are both stable under the natural action of $GL(V)$. Clearly, each $GL(V)$-orbit in Σ_{simp} or Σ_{solv} corresponds to an isomorphism class of Lie algebras.

Actually, two cases in Corollary 2 just correspond to two varieties in Theorem 3. The variety Σ_{simp} is a linear subspace of $\wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ defined by the condition $\mathrm{Tr}\,\mathrm{ad}\,X = 0$ $(X \in \mathfrak{g})$, i.e., Σ_{simp} is just the set of unimodular Lie algebras. The defining equation of the variety Σ_{solv} is given by $\dim\,[\mathfrak{g},\mathfrak{g}] \leq 2$, which is a cubic polynomial relation on c_{ij}^k. It is known that both varieties are of the same dimension 6.

§ 2. $GL(V)$-irreducible components of $S^p(\wedge^2 V^* \otimes V)^*$.

As stated above, the set of 3-dimensional Lie algebras is the union of two $GL(V)$-invariant irreducible varieties, and hence, their defining equations form $GL(V)$-invariant subspaces of the polynomial ring $\sum\, S^p(\wedge^2 V^* \otimes V)^*$. In this section, in order to describe several $GL(V)$-invariant concepts in Σ_{simp} and Σ_{solv}, including the representation theoretic meaning of their defining equations, we first give a list of irreducible decomposition of $S^p(\wedge^2 V^* \otimes V)^*$ and the generator of each component up to degree 3.

For simplicity, we fix a volume form ω of V as in the proof of Theorem 1, and identify two spaces $\wedge^2 V^* \otimes V$ and $V \otimes V$. Then, as $GL(V)$-modules, these spaces are isomorphic to each other by multiplying an invariant of $SL(V)$ corresponding to a suitable power of det g ($g \in GL(V)$). This argument also holds for polynomials on $\wedge^2 V^* \otimes V \cong V \otimes V$ for each fixed dgree. In the following, we only consider algebraic concepts determined by the vanishing (or by the ratio) of these polynomials, and hence, we may use the space $S^p(V \otimes V)^*$ instead of $S^p(\wedge^2 V^* \otimes V)^*$.

The explicit decomposition of $S^p(V \otimes V)^*$ is easily obtained by using Littlewood-Richardson's rule as in § 1, and the generator of each irreducible component can be calculated by the method stated in [3; p.115 $\sim$ 116]. (By this method, the highest element in S_λ automatically appears.) Most generators are expressed as a product of lower degree generators, and we marked the symbol "$*$" to the generators that cannot be expressed as a product form. These marked generators correspond to covariants (or invariants) in classical invariant theory (cf. [9], [16], [20]). In the following, we express c_{ij}^k as a matrix form: $c_{i1} = c_{23}^i$, $c_{i2} = c_{31}^i$, $c_{i3} = c_{12}^i$ ($i = 1 \sim 3$).

$p = 1 :\ S_2 + S_{11}$

$$\begin{aligned} * \, S_2 \ &: c_{11} \\ * \, S_{11} \ &: c_{12} - c_{21} \end{aligned}$$

$p = 2 :\ S_4 + S_{31} + 2\,S_{22} + S_{211}$

$$\begin{aligned} S_4 \ &: c_{11}^2 = S_2^2 \\ S_{31} \ &: c_{11}(c_{12} - c_{21}) = S_2 S_{11} \\ * \, S_{22_a} \ &: c_{11}c_{22} - c_{12}c_{21} \\ S_{22_b} \ &: (c_{12} - c_{21})^2 = S_{11}^2 \\ * \, S_{211} \ &: c_{11}c_{23} - c_{11}c_{32} + c_{12}c_{31} - c_{13}c_{21} \end{aligned}$$

$p = 3 :\ S_6 + S_{51} + 2\,S_{42} + S_{411} + 2\,S_{33} + 2\,S_{321} + 2\,S_{222}$

$$\begin{aligned} S_6 \ &: c_{11}^3 = S_2^3 \\ S_{51} \ &: c_{11}^2(c_{12} - c_{21}) = S_2^2\, S_{11} \\ S_{42_a} \ &: c_{11}(c_{12} - c_{21})^2 = S_2\, S_{11}^2 \\ S_{42_b} \ &: c_{11}(c_{11}c_{22} - c_{12}c_{21}) = S_2\, S_{22_a} \\ S_{411} \ &: c_{11}(c_{11}c_{23} - c_{11}c_{32} + c_{12}c_{31} - c_{13}c_{21}) = S_2\, S_{211} \\ S_{33_a} \ &: (c_{12} - c_{21})^3 = S_{11}^3 \\ S_{33_b} \ &: (c_{12} - c_{21})(c_{11}c_{22} - c_{12}c_{21}) = S_{11} S_{22_a} \\ * \, S_{321_a} &: \begin{vmatrix} c_{11} & c_{11} & c_{21} \\ c_{21} & c_{12} & c_{22} \\ c_{31} & c_{13} & c_{23} \end{vmatrix} \end{aligned}$$

$$S_{321_b}:\ (c_{12}-c_{21})(c_{11}c_{23}-c_{11}c_{32}+c_{12}c_{31}-c_{13}c_{21}) = S_{11}\,S_{211}$$

$$*\ S_{222_a}:\ \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

$$\begin{aligned} *\ S_{222_b}:\ & c_{11}(c_{23}-c_{32})^2+c_{22}(c_{13}-c_{31})^2+c_{33}(c_{12}-c_{21})^2 \\ & +(c_{12}-c_{21})(c_{23}c_{31}-c_{13}c_{32})+(c_{23}-c_{32})(c_{31}c_{12}-c_{21}c_{13}) \\ & +(c_{31}-c_{13})(c_{12}c_{23}-c_{32}c_{21}) \end{aligned}$$

Note that the dimension of the irreducible representation space corresponding to the partition $\{a, b, c\}$ is given by

$$\dim S_{abc} = \frac{1}{2}(a-b+1)(a-c+2)(b-c+1)$$

in the case $\dim V = 3$. In the following, for each irreducible component S_λ, we often express its generator by using the same letter S_λ. (If the multiplicity is greater than one, we distinguish these generators by the symbols S_{λ_a}, S_{λ_b}, etc.)

In viewing the above list, we know that there appear seven fundamental covariants (= essential generators marked by "$*$") up to degree 3. We conjecture that the polynomial ring $\sum S^p(V \otimes V)^*$ is generated by these seven fundamental covariants, namely, each generator of the component of $S^p(V \otimes V)^*$ is expressed as a product of these seven covariants. By direct calculations, we can show that this conjecture is correct up to degree 8. But it seems that the complete determination of covariants and their syzygies are hard to solve as in classical invariant theory, though it is an important and fundamental problem. (For the spaces S^2V, $\wedge^2 V$ and $V \oplus \wedge^2 V$, we know the complete answer to this problem. (cf. [1], [2], [8]. See also [4].)) If the above conjecture is correct, all $GL(V)$-invariant concepts in the space $V \otimes V$ ($V = \mathbf{C}^3$) can be described by these seven covariants.

We remark that these seven polynomials are not functionally independent on $V \otimes V$. In fact, these generators satisfy the following sextic identity:

$$S_2\,S_{11}^2\,S_{222_a} + S_{211}^2\,S_{22_a} + S_{321_a}^2 = S_2\,S_{22_a}\,S_{222_b} + S_{11}\,S_{211}\,S_{321_a}.$$

The space $S^6(V \otimes V)^*$ contains the irreducible components S_{642} with multiplicity seven. But there exist eight generators corresponding to S_{642}:

$$\begin{aligned} & S_2\,S_{11}^2\,S_{222_a},\quad S_2\,S_{11}^2\,S_{222_b},\quad S_2\,S_{22_a}\,S_{222_a},\quad S_2\,S_{22_a}\,S_{222_b}, \\ & S_{11}^2\,S_{211}^2,\quad S_{11}\,S_{211}\,S_{321_a},\quad S_{211}^2\,S_{22_a},\quad S_{321_a}^2. \end{aligned}$$

And the difference $8 - 7 = 1$ implies the existence of the above sextic identity. This is the unique non-trivial polynomial relation among seven fundamental covariants because the rank of the $(7,9)$-matrix $(\partial S_\lambda / \partial c_{ij})$ is 6 at a generic point of $V \otimes V$, where S_λ are the fundamental covariants.

§ 3. **Four covariants for 3-dimensional Lie algebras.**

Now, we impose the condition $S_{211} = 0$ corresponding to the Jacobi identity, i.e., we assume that the space V possesses a Lie algebra structure. (The expression $S_\lambda = 0$

implies that not only the generator itself, but all polynomials in the space S_λ vanish.) In this section, we give a geometric meaning defined by the fundamental covariants listed in § 2. But in advance, we first review a classification of 3-dimensional complex Lie algebras for later use (cf. [10], [14], [19]).

	non-trivial bracket operations
$L_0 = \mathbf{C}^3$	
$L_1 =$ Heisenberg	$[X_1, X_2] = X_3$
$L_2 = \mathfrak{aff}(1, \mathbf{C}) \oplus \mathbf{C}^1$	$[X_1, X_2] = X_2$
L_3	$[X_1, X_2] = X_2, \ \ [X_1, X_3] = X_2 + X_3$
$L_4(\alpha)$	$[X_1, X_2] = X_2, \ \ [X_1, X_3] = \alpha X_3$
$L_5 = \mathfrak{sl}(2, \mathbf{C})$	$[X_1, X_2] = X_2, \ \ [X_1, X_3] = -X_3, \ \ [X_2, X_3] = X_1$

For the Lie algebra $L_4(\alpha)$, it is known that $L_4(\alpha) \cong L_4(\alpha')$ if and only if $\alpha = \alpha'$ or $\alpha\alpha' = 1$. In addition, we have $L_4(0) = L_2$, and hence, we may assume $|\alpha| \geq 1$. Among them, we can easily check that $L_4(-1)$ is isomorphic to the complex Euclidean Lie algebra

$$\mathfrak{e}(2, \mathbf{C}) = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

At a first glance, the Lie algebra L_3 seems to be isolated from other Lie algebras. But this appearance is not correct, and it is continuously deformable from the Lie algebra $L_4(\alpha)$ as follows. We define a Lie algebra $M(\alpha)$ by

$$[X_1, X_2] = X_2, \ \ [X_1, X_3] = X_2 + \alpha\, X_3.$$

Then, we can easily show that

$$M(\alpha) \cong \begin{cases} L_4(\alpha) & (\alpha \neq 0, \ 1) \\ L_2 & (\alpha = 0) \\ L_3 & (\alpha = 1) \end{cases},$$

which implies that the Lie algebra L_3 is adjacent to $L_4(\alpha)$ in spite of its appearance.

Now, for each Lie algebra $\mathfrak{g}$, we substitute the value c_{ij}^k to seven fundamental covariants listed in § 2, where c_{ij}^k is the structure constant with respect to a generic basis of $\mathfrak{g}$. Then, it follows that the equalities

$$S_{211} = S_{321_a} = S_{222_b} = 0$$

hold for any 3-dimensional Lie algebra, and hence there remain four fundamental covariants S_2, S_{11}, S_{22_a} and S_{222_a}. The value for these covariants are summarized in the following table. (We assume that the complex number α satisfies $|\alpha| \geq 1, \alpha \neq \pm 1$.)

	S_2	S_{11}	S_{22_a}	S_{222_a}	dimension of $GL(V)$-orbit
L_0	0	0	0	0	0
L_1	$*$	0	0	0	3
L_2	$*$	$*$	0	0	5
L_3	$*$	$*$	$*$	0	5
$L_4(\alpha)$	$*$	$*$	$*$	0	5
$L_4(1)$	0	$*$	$*$	0	3
$L_4(-1)$	$*$	0	$*$	0	5
L_5	$*$	0	$*$	$*$	6

Here, the symbol "0" implies that the generator S_λ vanishes with respect to any basis of $\mathfrak{g}$, and the symbol "$*$" implies that S_λ does not vanish for a generic basis. (In this table, we also exhibit the dimension of each $GL(V)$-orbit.)

We summarize the $GL(V)$-invariant concepts defined by these covariants as follows. (As stated before, we express V as $\mathfrak{g}$.)

Proposition 4. (1) *The condition* $S_2 = 0$ *implies that there exists* $\varphi \in \mathfrak{g}^*$ *such that* $[X, Y] = \varphi(X)Y - \varphi(Y)X$. *These Lie algebras constitute a* 3*-dimensional linear subspace contained in the variety* Σ_{solv}.

(2) *The condition* $S_{11} = 0$ *implies that* $\mathfrak{g}$ *is unimodular, and this is the defining equation of the variety* Σ_{simp}. (*We remark that the condition* $S_{11} = 0$ *automatically implies* $S_{211} = 0$. *Of course, combined linear conditions* $S_2 = S_{11} = 0$ *imply that* $\mathfrak{g}$ *is abelian.*)

(3) *The condition* $S_{22_a} = 0$ *holds if and only if* $\dim\,[\mathfrak{g}, \mathfrak{g}] \leq 1$.

(4) *The condition* $S_{222_a} = 0$ *holds if and only if* $\dim\,[\mathfrak{g}, \mathfrak{g}] \leq 2$, *and this condition is also equivalent to the solvability of* $\mathfrak{g}$. *In particular, the defining equations of* Σ_{solv} *are given by* $S_{211} = S_{222_a} = 0$.

(5) *Nilpotent Lie algebras are characterized by two conditions* $S_{11} = S_{22_a} = 0$, *i.e., a* 3*-dimensional Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if it is unimodular and* $\dim\,[\mathfrak{g}, \mathfrak{g}] \leq 1$. (*Actually, it is* 2*-step nilpotent.*)

Proof. The results in (1) and (2) follow immediately from the explicit irreducible decomposition $(V \otimes V)^* = S_2 \oplus S_{11}$. The statements in (3) and (4) follow from the fact that S_{22_a} and S_{222_a} are the principal minor and the determinant of the matrix (c_{ij}), respectively (cf. [1], [2]). The result (5) can be easily obtained by using the classification stated above. q.e.d.

The explicit description of the $GL(V)$-invariant varieties determined by these conditions will be described in detail in the next section.

§ 4. Varieties of 3-dimensional Lie algebras.

In this section, we study $GL(V)$-invariant varieties consisting of 3-dimensionl Lie algebras and the orbit structures of them, in terms of four fundamental covariants listed up in § 3.

In the following, we denote by $\mathcal{O}(L_k)$ the $GL(V)$-orbit of the Lie algebra L_k, and by $\overline{\mathcal{O}(L_k)}$ its Zariski closure. The orbit decompositions in the following proposition are more or less well known for many Lie algebraists. But, these decompositions are easily proved by using the results stated in the previous sections. For example, we have $L_4(1) \notin \overline{\mathcal{O}(L_2)}$ because $S_{22_a} \neq 0$ for $L_4(1)$. (Remind that the defining equations of Σ_{simp} and Σ_{solv} are $S_{11} = 0$ and $S_{211} = S_{222_a} = 0$, respectively. The dimension of each orbit is stated in § 3.)

Proposition 5. (1) *The varieties Σ_{simp} and Σ_{solv} are the union of the following $GL(V)$-orbits*:

$$\begin{aligned}\Sigma_{simp} &= \mathcal{O}(L_5) \cup \mathcal{O}(L_4(-1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0),\\ \Sigma_{solv} &= \cup_\alpha \mathcal{O}(L_4(\alpha)) \cup \mathcal{O}(L_3) \cup \mathcal{O}(L_2) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0) \quad (|\alpha| \geq 1).\end{aligned}$$

And hence, we have $\Sigma_{simp} \cap \Sigma_{solv} = \mathcal{O}(L_4(-1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0)$.

(2) *The $GL(V)$-orbit decompositions of $\overline{\mathcal{O}(L_k)}$ are given by*

$$\begin{aligned}\overline{\mathcal{O}(L_0)} &= \mathcal{O}(L_0) = \{0\},\\ \overline{\mathcal{O}(L_1)} &= \mathcal{O}(L_1) \cup \mathcal{O}(L_0),\\ \overline{\mathcal{O}(L_2)} &= \mathcal{O}(L_2) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0),\\ \overline{\mathcal{O}(L_3)} &= \mathcal{O}(L_3) \cup \mathcal{O}(L_4(1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0),\\ \overline{\mathcal{O}(L_4(\alpha))} &= \mathcal{O}(L_4(\alpha)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0), \quad (|\alpha| \geq 1, \alpha \neq \pm 1),\\ \overline{\mathcal{O}(L_4(1))} &= \mathcal{O}(L_4(1)) \cup \mathcal{O}(L_0),\\ \overline{\mathcal{O}(L_4(-1))} &= \mathcal{O}(L_4(-1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0) = \Sigma_{simp} \cap \Sigma_{solv},\\ \overline{\mathcal{O}(L_5)} &= \mathcal{O}(L_5) \cup \mathcal{O}(L_4(-1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0) = \Sigma_{simp}.\end{aligned}$$

A generic element of Σ_{simp} is represented by L_5 $(= \mathfrak{sl}(2, \mathbf{C}))$ and the Lie algebras $L_4(-1)$, L_1, L_0 are degenerations of L_5. (For the concept "degeneration", see [12], [19].) These Lie algebras are distinguished in Σ_{simp} by the rank of the matrix (c_{ij}), as stated in [15]. The other variety Σ_{solv} does not admit a dense orbit, in contrast with Σ_{simp}, and it mainly consists of infinite family of 5-dimensional orbits, which implies the existence of a 1-dimensional moduli space. (Remind that dim Σ_{simp} = dim $\Sigma_{solv} = 6$.)

Concerning this moduli space, an invariant $\chi(\mathfrak{g}) \in \mathbf{C} \cup \{\infty\}$ was introduced in the papers [15], [22]. (In the following, we mainly use the notation in [22]. The invariant J defined in [15] is related to $\chi(\mathfrak{g})$ in [22] by the equality $\chi(\mathfrak{g}) = 2/(1 - J)$.) This invariant is quite useful in classifying 3-dimensional Lie algebras in Σ_{solv}, and roughly speaking, it gives a coordinate of the above moduli space. In our viewpoint, this invariant $\chi(\mathfrak{g})$ can be expressed as the ratio of two covariants belonging to the same partition $\{22\}$:

$$(*) \qquad \chi(\mathfrak{g}) = \frac{S_{22_b}}{S_{22_a}} = \frac{S_{11}^2}{S_{22_a}} = \frac{(c_{12} - c_{21})^2}{c_{11}c_{22} - c_{12}c_{21}}.$$

This invariant is well-defined for all 3-dimensional Lie algebras except nilpotent ones (= L_0 and L_1). (Remind that nilpotent Lie algebras are characterized by two conditions

$S_{11} = S_{22_a} = 0$.) We can easily show that the value at the right end of $(*)$ does not depend on the choice of a basis of $\mathfrak{g}$. And hence in case $c_{12} - c_{21} = c_{11}c_{22} - c_{12}c_{21} = 0$ for some basis, we have only to change it such that $\chi(\mathfrak{g})$ has a definite value in order to calculate $\chi(\mathfrak{g})$. Of course, we can also calculate this invariant by using the definition

$$J = \frac{\mathrm{Tr}\ (\mathrm{ad}\ X)^2}{(\mathrm{Tr}\ \mathrm{ad}\ X)^2} = 1 - \frac{2}{\chi(\mathfrak{g})}$$

in [15], which is independent of the choice of generic $X \in \mathfrak{g}$. Essentially, the invariant $\chi(\mathfrak{g})$ is determined by the ratio of two non-zero eigenvalues of ad X. In fact, denoting the eigenvalues of ad X by 0, α, β, we have the equality

$$\chi(\mathfrak{g}) = \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + 2.$$

The explicit value of $\chi(\mathfrak{g})$ for each Lie algebra is given as follows:

	$\chi(\mathfrak{g})$
L_2	∞
L_3	4
$L_4(\alpha)$	$\frac{(\alpha+1)^2}{\alpha}$
L_5	0

As we listed in § 3, there are three types of 5-dimensional $GL(V)$-orbits in Σ_{solv} : $\mathcal{O}(L_2)$, $\mathcal{O}(L_3)$ and $\mathcal{O}(L_4(\alpha))$ ($|\alpha| \geq 1, \alpha \neq 1$). For these Lie algebras, we have the following proposition, indicating that the invariant $\chi(\mathfrak{g})$ serves as a coordinate of the moduli space of these Lie algebras. This proposition is almost equivalent to Theorem in [21], or the normal form stated in [15; p.23] and [22; Theorem 2], though the unimodular Lie algebra $L_4(-1)$ is excluded there.

Proposition 6. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two elements of $\mathcal{O}(L_2) \cup \mathcal{O}(L_3) \cup \mathcal{O}(L_4(\alpha))$ ($|\alpha| \geq 1$, $\alpha \neq 1$) $= \Sigma_{solv} \setminus \mathcal{O}(L_4(1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0)$. Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$ if and only if $\chi(\mathfrak{g}) = \chi(\mathfrak{g}')$.*

This proposition follows immediately from the fact that the condition $(\alpha+1)^2/\alpha = (\alpha'+1)^2/\alpha'$ is equivalent to $\alpha = \alpha'$ or $\alpha\alpha' = 1$. (We remark that $\chi(L_3) = \chi(L_4(1))$ and the invariant $\chi(\mathfrak{g})$ does not distinguish two Lie algebras L_3 and $L_4(1)$.)

The set of Lie algebras appearing in this proposition is dense in Σ_{solv}, and the remaining Lie algebras $L_4(1)$, L_1 and L_0 in Σ_{solv} are the degenerations of them.

It should be noted that in terms of this invariant $\chi(\mathfrak{g})$, Lie algebras in $\mathcal{O}(L_2) \cup \mathcal{O}(L_3) \cup \mathcal{O}(L_4(\alpha))$ ($|\alpha| \geq 1$, $\alpha \neq \pm 1$) $= \Sigma_{solv} \setminus \mathcal{O}(L_4(\pm 1)) \cup \mathcal{O}(L_1) \cup \mathcal{O}(L_0)$ are expressed as

$$[e_1,\, e_2] = 0, \quad [e_2,\, e_3] = e_1, \quad [e_3,\, e_1] = -e_1 + \frac{1}{\chi(\mathfrak{g})} e_2$$

for a suitable basis e_i of $\mathfrak{g}$, as stated in [22]. (The sign of $[e_2, e_3]$ in [22; Theorem 2] should be corrected to the above form, as Professor Umehara kindly teach this fact to the author.)

Combining these results, we can give a simple description of the set of 3-dimensional Lie algebras from $GL(V)$-invariant viewpoint as follows: The variety Σ_{solv} is the closure of the family of 5-dimensional $GL(V)$-orbits, which are continuously deformable to each other. Any two closures of these orbits have a common intersection $\overline{\mathcal{O}(L_1)} = \mathcal{O}(L_1) \cup \mathcal{O}(L_0)$. Among these 5-dimensional orbits, $\mathcal{O}(L_3)$ has a special feature, containing an exceptional orbit $\mathcal{O}(L_4(1))$ in its closure. Another exceptional 5-dimensional orbit is $\mathcal{O}(L_4(-1))$, which is contained in the other variety Σ_{simp}. Actually, the intersection $\Sigma_{simp} \cap \Sigma_{solv}$ just coincides with the closure $\overline{\mathcal{O}(L_4(-1))}$, as stated in Proposition 5.

Finally, summarizing these results, we give an algorithm to determine the isomorphism classes of 3-dimensional Lie algebras, which is quite useful in actual determination:

$$
\begin{array}{ccl}
\text{START} & & \\
\downarrow & & \\
\text{abel} & \xrightarrow{Y} & L_0 = \mathbf{C}^3 \\
{\scriptstyle N}\downarrow & & \\
\dim\, [\mathfrak{g}, \mathfrak{g}] = 3 & \xrightarrow{Y} & L_5 = \mathfrak{sl}(2, \mathbf{C}) \\
{\scriptstyle N}\downarrow & & \\
S_2 = 0 & \xrightarrow{Y} & L_4(1) \\
{\scriptstyle N}\downarrow & & \\
\dim\, [\mathfrak{g}, \mathfrak{g}] = 1 \ \& \ \text{unimodular} & \xrightarrow{Y} & L_1 = \text{Heisenberg} \\
{\scriptstyle N}\downarrow & & \\
\left\{ \begin{array}{ll} \chi(\mathfrak{g}) & = \infty \\ \chi(\mathfrak{g}) & = 4 \\ \chi(\mathfrak{g}) & = 0 \\ \chi(\mathfrak{g}) & = (\alpha+1)^2/\alpha \end{array} \right. & \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} & \begin{array}{l} L_2 = \mathfrak{aff}(1, \mathbf{C}) \oplus \mathbf{C}^1 \\ L_3 \\ L_4(-1) = \mathfrak{e}(2, \mathbf{C}) \\ L_4(\alpha) \quad (|\alpha| \geq 1, \alpha \neq \pm 1) \end{array}
\end{array}
$$

Remind that the condition $S_2 = 0$ is equivalent to $[X, Y] = \varphi(X)Y - \varphi(Y)X$ for some $\varphi \in \mathfrak{g}^*$, which is also equivalent to the condition $c_{ij} + c_{ji} = 0$ (i.e., Lie algebras corresponding to $\mathfrak{k}$ in [22]).

Viewing this algorithm, we once more emphasize that all isomorphism classes are completely characterized in terms of four covariants S_2, S_{11}, S_{22_a} and S_{222_a}. We may say that these covariants (including the invariant $\chi(\mathfrak{g}) = S_{11}^2/S_{22_a}$) serve as a complete measure in describing the structure of 3-dimensional Lie algebras, which shows the effectiveness of our $GL(V)$-invariant viewpoint stated in Introduction.

§ 5. **Final remarks.**

(1) As stated in the previous sections, two varieties Σ_{simp} and Σ_{solv} admit their own defining equations in addition to the Jacobi identity. This fact implies that the quadratic polynomials corresponding to the Jacobi identity essentially contain another higher or curiously lower degree polynomial relations in an implicit form. And such a phenomenon

usually occurs in multi-tensor spaces (cf. [4], [5]). For 3-dimensional Lie algebras, this phenomenon follows from the identity in Theorem 1. But for higher dimensional cases, it seems that such a fundamental identity does not exist, and it is desirable to investigate another method to explain such a phenomenon.

(2) In the 3-dimensional case, four covariants are sufficient to describe Lie algebra structures on V. To apply our method to higher dimensional cases, we must know the $GL(V)$-irreducible decomposition of the polynomial ring $\sum S^p(\wedge^2 V^* \otimes V)^*$ at first. But this is a quite difficult problem in representation theory, related to "plethysm" and "3-tensor spaces". (As for the concept "plethysm", see [17]. The space $\wedge^2 V^* \otimes V$ may be considered as a 3-tensor space because V and V^* symbolically appear three times in $\wedge^2 V^* \otimes V$. See [4].) For example, in the 4-dimensional case, we know by direct calculations that the numbers of fundamental covariants in $S^p(\wedge^2 V^* \otimes V)^*$ ($p = 1 \sim 4$) are given by 2, 5, 14 and 28, respectively (cf. [6]). For sufficiently high dimensional case, these numbers become 2, 7, 40 and 255, respectively, and it seems almost impossible to obtain a formula for general p. We may say that we are just encountering the usual difficulty peculiar to classical invariant theory.

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