Solutions and almost solutions of the Gauss equation of $SU(3)/SO(3)$

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Abstract

We give new solutions and almost solutions of the Gauss equation of the Riemannian symmetric spaces $SU(3)/SO(3)$ and its non-compact dual in codimensions 4 and 5, which improve the previously known estimates on the codimension. We also give experimental estimates on the infimum of the norm $\|\gamma_r(\alpha)\pm R\|$ for each codimension r, where R is the curvature of $SU(3)/SO(3)$, and α runs all over the space of second fundamental forms.

Key words: local isometric imbedding, Riemannian symmetric space, Gauss equation, curvature.

Introduction.

The explicit determination of the least dimensional Euclidean space into which a given Riemannian manifold M can be (locally) isometrically immersed is one of the classical fundamental problem in Riemannian geometry. But unfortunately, only a few explicit results are known at present concerning this problem. Even restricting to local isometric imbeddings of Riemannian symmetric spaces, the least dimensional Euclidean spaces are determined only for the spaces $\mathbf{R}^n$, S^n, H^n (the hyperbolic space), [CI] $Sp(n)/U(n)$ and $Sp(n)$ (cf. [5], [11], [12]).

To solve this local isometric imbedding problem, it is necessary to determine the least codimension where the Gauss equation admits a solution for a given Riemannian manifold, as a first step problem. In fact, the Gauss equation appears as a first obstruction to the existence of solutions of the differential equations of local isometric imbeddings. But, solving the Gauss equation is in general a hard algebraic problem because it is equivalent to solve a system of real quadratic equations with many equations and many variables, and at present, a little results are obtained for only a special class of Riemannian manifolds. For example, we know that the least codimension where the Gauss equation of complex projective space $M^4 = P^2(\mathbf{C})$ admits a solution is equal to 3 (cf. [2; p.128, 132]). For other results, see the references at the end of this paper.

Now in this paper, we consider the case of 5-dimensional Riemannian symmetric spaces $SU(3)/SO(3)$ and its non-compact dual, and give a family of solutions of the Gauss equation in codimension 5, which improves the previously known results. In addition, we give "almost solutions" in codimension 4 for both spaces. Here, "almost solutions" imply a family of second fundamental forms α_t depending on a parameter t such that the curvature determined by α_t in terms of the Gauss equation converges to a given curvature, though α_t itself diverges as t goes to zero. It is already known that $SU(3)/SO(3)$ is globally isometrically imbedded into the Euclidean space with codimension 7, and admits a solution of the Gauss equation in codimension 6 (cf. [17]), and this is the best results previously known. Hence, our results in this paper improve the estimates on the codimension where the Gauss equation admits a solution. But in spite of these improvements, it is still an open question which is the "least" codimension where $SU(3)/SO(3)$ (or its dual) is locally isometrically immersed. It is our next problem whether the second fundamental forms we construct in this paper are the actual second fundamental forms for some local isometric imbeddings of $SU(3)/SO(3)$ (or its dual) into $\mathbf{R}^{10}$.

To obtain the results of this paper, computational experiments by the software Mathematica are quite useful in finding solutions and almost solutions. At the end of this paper, we add some numerical estimates on the infimum of the norm $\| \gamma_r(\alpha) \pm R \|$ for each codimension r, where R is the curvature of $SU(3)/SO(3)$, and α runs all over the space of second fundamental forms. These results suggest that the least codimension such that the Gauss equation admits a solution is 4 or 5 for both spaces.

§ 1. Curvature of $SU(3)/SO(3)$ and the main theorem.

We first write down the curvature of the space $SU(3)/SO(3)$. We put $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{k} = \mathfrak{o}(3)$ and

$$\mathfrak{m} = \left\{ i \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & c \end{pmatrix} \middle| \begin{array}{l} a,\ b,\ c,\ p,\ q,\ r \in \mathbf{R}, \\ a+b+c=0 \end{array} \right\}.$$

Then, we have the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $SU(3)/SO(3)$, and the space $\mathfrak{m}$ may be considered as the tangent space of $SU(3)/SO(3)$ at the origin. The bi-invariant metric of $SU(3)/SO(3)$ is given by $\langle X, Y \rangle = -1/2 \cdot \mathrm{Tr}\, XY$ (X, $Y \in \mathfrak{m}$), and the curvature of this metric is given by

$$R(X, Y, Z, W) = -1/2 \cdot \mathrm{Tr}\, [X, Y][Z, W].$$

We fix the orthonormal basis $\{X_1, \cdots, X_5\}$ of $\mathfrak{m}$ by

$$X_1 = \begin{pmatrix} i & & \\ & 0 & \\ & & -i \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} i & & \\ & -2i & \\ & & i \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i & \\ i & 0 & \\ & & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & & i \\ & 0 & \\ i & & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & & \\ & 0 & i \\ & i & 0 \end{pmatrix}.$$

Then the components of the curvature tensor are explicitly given by

$$\begin{aligned}
&R_{1313} = R_{1345} = R_{1515} = R_{1534} = R_{3434} = R_{3535} = R_{4545} = 1, \\
&R_{1323} = R_{2345} = \sqrt{3}, \quad R_{1525} = R_{2534} = -\sqrt{3}, \\
&R_{1435} = 2, \qquad\qquad\quad R_{2323} = R_{2525} = 3, \\
&R_{1414} = 4,
\end{aligned}$$

and other $R_{ijkl} = 0$ except for the components obtained by the symmetric property

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk},$$

where R_{ijkl} is one of the above (cf. [2; p.129]). The curvature of the non-compact dual space of $SU(3)/SO(3)$ is given by $-R$.

Let K be the space of curvature like tensors on $\mathfrak{m}$, i.e.,

$$K = \left\{ R \in \wedge^2\mathfrak{m}^* \otimes \wedge^2\mathfrak{m}^* \mid \mathfrak{S}_{X,Y,Z}\, R(X,Y,Z,W) = 0 \right\}.$$

We define a quadratic map

$$\gamma_r : S^2\mathfrak{m}^* \otimes \mathbf{R}^r \longrightarrow K$$

by

$$\gamma_r(\alpha)(X,Y,Z,W) = \langle \alpha(X,Z), \alpha(Y,W) \rangle - \langle \alpha(X,W), \alpha(Y,Z) \rangle$$

for $\alpha \in S^2\mathfrak{m}^* \otimes \mathbf{R}^r$, X, Y, Z, $W \in \mathfrak{m}$, and denote the image of this map by $\mathrm{Im}\ \gamma_r$. The image $\mathrm{Im}\ \gamma_r$ is not in general closed in K (cf. [2; p.134]), and we denote its Zariski closure by $\overline{\mathrm{Im}\ \gamma_r}$. The space $S^2\mathfrak{m}^* \otimes \mathbf{R}^r$ indicates the space of pointwise second fundamental forms of submanifolds with codimension r. Then, the Gauss equation of $SU(3)/SO(3)$ (and its dual) with codimension r is expressed as

$$\gamma_r(\alpha) = \pm R,$$

where R is the curvature of $SU(3)/SO(3)$ stated above. Clearly, the Gauss equation admits a solution if and only if $R \in \mathrm{Im}\ \gamma_r$ (or $-R \in \mathrm{Im}\ \gamma_r$, in the dual case). In particular, if we can prove $R \notin \mathrm{Im}\ \gamma_r$ (or $-R \notin \mathrm{Im}\ \gamma_r$) for some r, then it follows that $SU(3)/SO(3)$ (or its dual) cannot be isometrically immersed into the Euclidean space with codimension r even locally.

Concerning the existence or non-existence of the solutions of the Gauss equation of $SU(3)/SO(3)$ and its non-compact dual, we have the following results, which is the main theorem of this paper.

Theorem. *Let R and $-R$ be the curvatures of $SU(3)/SO(3)$ and its non-compact dual space, respectively. Then, we have*
(1) $R,\ -R \notin \mathrm{Im}\ \gamma_2$.
(2) $R,\ -R \in \overline{\mathrm{Im}\ \gamma_4}$.
(3) $R,\ -R \in \mathrm{Im}\ \gamma_5$.

Note that the space $SU(3)/SO(3)$ is globally isometrically imbedded into the Euclidean space with codimension 7 (cf. [17]), and it is already known that $R \in \mathrm{Im}\ \gamma_6$ because the 1-dimensionally extended space $U(3)/SO(3)$ admits a solution of the Gauss equation in codimension 6. Previously known results on the existence of solutions of the Gauss equation are exhausted by these facts, and hence the above theorem improves the estimates on the codimension where the Gauss equation admits a solution. But at present, we do not know whether R (or $-R$) is contained in $\mathrm{Im}\ \gamma_3$ or $\mathrm{Im}\ \gamma_4$. The above statement (2) indicates that the Gauss equation admits an "almost solution" in codimension 4, and it implies that the solvability of the Gauss equation in codimension 4 is a quite delicate problem.

§ 2. **Proof of Theorem.**

The result (1) is already proved in [2; p.128-129], [4; p.19]. We first construct solutions of the Gauss equation of $SU(3)/SO(3)$ in codimension 5. In general, the Gauss equation of the case $M^5 \subset \mathbf{R}^{10}$ is equivalent to the system of 50 quadratic polynomial equations with 75 variables, and it is almost impossible to solve it directly. We consider the situation where the second fundamental form $\alpha = (\alpha_1, \cdots, \alpha_5) \in S^2\mathfrak{m}^* \otimes \mathbf{R}^5$ is expressed in the following restricted form:

$$\alpha_1 = \begin{pmatrix} a_1 & a_6 & & & \\ a_6 & a_2 & & & \\ & & a_3 & & \\ & & & a_4 & \\ & & & & a_5 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} b_1 & b_6 & & & \\ b_6 & b_2 & & & \\ & & b_3 & & \\ & & & b_4 & \\ & & & & b_5 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & & c_1 & & \\ & 0 & c_2 & & \\ c_1 & c_2 & 0 & & \\ & & & 0 & c_3 \\ & & & c_3 & 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 0 & & & & d_1 \\ & 0 & & & d_2 \\ & & 0 & d_3 & \\ & & d_3 & 0 & \\ d_1 & d_2 & & & 0 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} 0 & & & & \\ & 0 & & e_1 & \\ & & 0 & & e_2 \\ & e_1 & & 0 & \\ & & e_2 & & 0 \end{pmatrix}.$$

Then, α satisfies the Gauss equation $\gamma_5(\alpha) = R$ if and only if it satisfies the following system of quadratic equation:

$$
\begin{aligned}
&a_1a_2 + b_1b_2 = a_6^2 + b_6^2, && c_1c_3 = 1,\\
&a_1a_3 + b_1b_3 = c_1^2 + 1, && c_2c_3 = d_2d_3,\\
&a_1a_4 + b_1b_4 = 4, && d_1d_3 = -1,\\
&a_1a_5 + b_1b_5 = d_1^2 + 1, && d_2d_3 = e_1e_2 - \sqrt{3},\\
&a_2a_3 + b_2b_3 = c_2^2 + 3, &&\\
&a_2a_4 + b_2b_4 = e_1^2, &&\\
&a_2a_5 + b_2b_5 = d_2^2 + 3, &&\\
&a_3a_4 + b_3b_4 = d_3^2 + 1, &&\\
&a_3a_5 + b_3b_5 = e_2^2 + 1, &&\\
&a_3a_6 + b_3b_6 = c_1c_2 + \sqrt{3}, &&\\
&a_4a_5 + b_4b_5 = c_3^2 + 1, &&\\
&a_4a_6 + b_4b_6 = 0, &&\\
&a_5a_6 + b_5b_6 = d_1d_2 - \sqrt{3}. &&
\end{aligned}
$$

This system of equations admits a family of solutions

$$
\begin{aligned}
&a_1 = a, && b_1 = b, &&\\
&a_2 = -\frac{bd}{a}, && b_2 = d, &&\\
&a_3 = a_5 = -\frac{a(3b - c^2d)}{c^2d(c^2+1)}, && b_3 = b_5 = \frac{3a^2 + bc^2d}{c^2d(c^2+1)}, &&\\
&a_4 = \frac{12a^2 - bd(c^2-1)^2}{3a(c^2+1)^2}, && b_4 = \frac{12b + d(c^2-1)^2}{3(c^2+1)^2}, &&\\
&a_6 = 0, && b_6 = 0, &&\\
&c_1 = c, && d_1 = c, && e_1 = \frac{\sqrt{3}\,d(c^2-1)}{3a},\\
&c_2 = -\frac{\sqrt{3}}{c}, && d_2 = \frac{\sqrt{3}}{c}, && e_2 = \frac{3a}{c^2d},\\
&c_3 = \frac{1}{c}, && d_3 = -\frac{1}{c}, &&
\end{aligned}
$$

where a, b, c, d are real numbers satisfying the conditions

$$a^2 + b^2 = (c^2+1)^2, \quad a \neq 0, \quad c \neq 0, \quad d \neq 0.$$

Next, we construct almost solutions of $SU(3)/SO(3)$ in codimension 4. We put $\alpha_5 = 0$ (i.e., $e_1 = e_2 = 0$) in the above notations. Then, we have $\alpha \in S^2\mathfrak{m}^* \otimes \mathbf{R}^4$. In addition, we set $c = 1$ in the above solutions of the Gauss equation. Then we have

$$\begin{aligned}
&a_1 = a_4 = a, && b_1 = b_4 = b,\\
&a_2 = -\frac{bd}{a}, && b_2 = d,\\
&a_3 = a_5 = -\frac{a(3b-d)}{2d}, && b_3 = b_5 = \frac{3a^2+bd}{2d},\\
&a_6 = 0, && b_6 = 0,\\
&c_1 = 1, && d_1 = 1,\\
&c_2 = -\sqrt{3}, && d_2 = \sqrt{3},\\
&c_3 = 1, && d_3 = -1,
\end{aligned}$$

where a, b, d are real numbers satisfying the conditions

$$a^2 + b^2 = 4, \quad a \neq 0, \quad d \neq 0.$$

In this situation, we have

$$(\gamma_4(\alpha) - R)_{3535} = \left(\frac{3a}{d}\right)^2,$$

and remaining components of $\gamma_4(\alpha) - R \in K$ are all zero. Hence, by putting $a \to 0$, we have $\gamma_4(\alpha) \to R$, which implies $R \in \overline{\mathrm{Im}\ \gamma_4}$.

Next, we consider the non-compact dual of $SU(3)/SO(3)$. We construct a family of solutions of the Gauss equation in codimension 5. We use the same notations as in the case of $SU(3)/SO(3)$ with codimension 5. Then $\alpha = (\alpha_1, \cdots, \alpha_5)$ is the solution of the Gauss equation $\gamma_5(\alpha) + R = 0$ if and only if

$$\begin{aligned}
&a_1a_2 + b_1b_2 = a_6^2 + b_6^2, && c_1c_3 = -1,\\
&a_1a_3 + b_1b_3 = c_1^2 - 1, && c_2c_3 = d_2d_3,\\
&a_1a_4 + b_1b_4 = -4, && d_1d_3 = 1,\\
&a_1a_5 + b_1b_5 = d_1^2 - 1, && d_2d_3 = e_1e_2 + \sqrt{3},\\
&a_2a_3 + b_2b_3 = c_2^2 - 3,\\
&a_2a_4 + b_2b_4 = e_1^2,\\
&a_2a_5 + b_2b_5 = d_2^2 - 3,\\
&a_3a_4 + b_3b_4 = d_3^2 - 1,\\
&a_3a_5 + b_3b_5 = e_2^2 - 1,\\
&a_3a_6 + b_3b_6 = c_1c_2 - \sqrt{3},\\
&a_4a_5 + b_4b_5 = c_3^2 - 1,\\
&a_4a_6 + b_4b_6 = 0,\\
&a_5a_6 + b_5b_6 = d_1d_2 + \sqrt{3}.
\end{aligned}$$

This system admits a family of solutions

$$
\begin{aligned}
&a_1 = a, && b_1 = b, && \\
&a_2 = -\frac{bd}{a}, && b_2 = d, && \\
&a_3 = a_5 = \frac{a(3b + c^2 d)(c^2 - 1)}{c^2 d(a^2 + b^2)}, && b_3 = b_5 = -\frac{(3a^2 - bc^2 d)(c^2 - 1)}{c^2 d(a^2 + b^2)}, && \\
&a_4 = -\frac{4a(3b + c^2 d)}{c^2 d(a^2 + b^2)}, && b_4 = \frac{4(3a^2 - bc^2 d)}{c^2 d(a^2 + b^2)}, && \\
&a_6 = 0, && b_6 = 0, && \\
&c_1 = c, && d_1 = c, && e_1 = -\frac{2\sqrt{3}}{c}, \\
&c_2 = \frac{\sqrt{3}}{c}, && d_2 = -\frac{\sqrt{3}}{c}, && e_2 = \frac{c^2 + 1}{2c}, \\
&c_3 = -\frac{1}{c}, && d_3 = \frac{1}{c}, &&
\end{aligned}
$$

where a, b, c, d are real numbers satisfying the conditions

$$a^2 + b^2 > 4c^2, \quad a \neq 0, \quad c \neq 0, \quad d = \pm\frac{6a}{c\sqrt{a^2 + b^2 - 4c^2}}.$$

Next, we construct almost solutions in codimension 4. In the above notations, we assume $\alpha_5 = 0$ (i.e., $e_1 = e_2 = 0$), and put

$$
\begin{aligned}
&a_1 = \frac{4a}{a^2 - 1}, && b_1 = \frac{b}{t}, && c_1 = -\frac{1}{a}, \\
&a_2 = 0, && b_2 = \frac{3b(a^2 - 1)^2}{t(a^2 + 1)^2}, && c_2 = \frac{\sqrt{3}}{a}, \\
&a_3 = \frac{1}{a}, && b_3 = -\frac{t(a^2 + 1)^2}{a^2 b(a^2 - 1)}, && c_3 = a, \\
&a_4 = -\frac{a^2 - 1}{a}, && b_4 = 0, && d_1 = a, \\
&a_5 = -a, && b_5 = \frac{t(a^2 + 1)^2}{b(a^2 - 1)}, && d_2 = \sqrt{3}\,a, \\
&a_6 = t, && b_6 = \frac{\sqrt{3}\,b(a^2 - 1)}{t(a^2 + 1)}, && d_3 = \frac{1}{a},
\end{aligned}
$$

where a, b and t are real numbers satisfying the conditions

$$a \neq 0, \pm 1, \quad b \neq 0, \quad t \neq 0.$$

Then, we have

$$
\begin{aligned}
&(\gamma_4(\alpha) + R)_{1212} = -t^2, && (\gamma_4(\alpha) + R)_{1525} = -at, \\
&(\gamma_4(\alpha) + R)_{1323} = \frac{t}{a}, && (\gamma_4(\alpha) + R)_{3535} = -\frac{t^2(a^2 + 1)^4}{a^2 b^2 (a^2 - 1)^2}, \\
&(\gamma_4(\alpha) + R)_{1424} = -\frac{t(a^2 - 1)}{a}, &&
\end{aligned}
$$

and remaining components of $\gamma_4(\alpha)+R$ are all zero. Hence, by putting $t \to 0$, we have $\gamma_4(\alpha) \to -R$, and hence we have $-R \in \overline{\mathrm{Im}\,\gamma_4}$. q.e.d.

§ 3. **Remarks.**

(1) The space $\mathfrak{m}^*$ is a 5-dimensional irreducible representation space of the isotropy group of $SU(3)/SO(3)$. Its symmetric 2-product $S^2\mathfrak{m}^*$ is decomposed into three $SO(3)$-irreducible components with dimensions 9, 5 and 1, respectively. Among them, the 5-dimensional irreducible subspace of $S^2\mathfrak{m}^*$ consists of matrices of the form

$$\begin{pmatrix} 2p_1 & 4p_2 & \sqrt{3}\,p_3 & 0 & \sqrt{3}\,p_4 \\ 4p_2 & -2p_1 & -p_3 & 2p_5 & p_4 \\ \sqrt{3}\,p_3 & -p_3 & -p_1+\sqrt{3}\,p_2 & -\sqrt{3}\,p_4 & \sqrt{3}\,p_5 \\ 0 & 2p_5 & -\sqrt{3}\,p_4 & 2p_1 & \sqrt{3}\,p_3 \\ \sqrt{3}\,p_4 & p_4 & \sqrt{3}\,p_5 & \sqrt{3}\,p_3 & -p_1-\sqrt{3}\,p_2 \end{pmatrix}, \quad p_i \in \mathbf{R},$$

where we express the element of $S^2\mathfrak{m}^*$ as a symmetric matrix by using the orthonormal basis $X_1, \cdots, X_5$ of $\mathfrak{m}$ which we defined before. Clearly, the second fundamental forms α which we constructed in §2 have a strong resemblance to this 5-dimensional irreducible subspace if we ignore the coefficients of p_i. We do not know whether we can construct a low codimensional solution of the Gauss equation for general $SU(m)/SO(m)$ $(m \geq 4)$ by applying the similar principle to some $SO(m)$-irreducible subspace of $S^2\mathfrak{m}^*$. (For general m, it is already known that the space $SU(m)/SO(m)$ is globally isometrically imbedded into the Euclidean space with codimension $1/2\cdot(m^2+m+2)$, but does not admit a solution of the Gauss equation in codimension $1/2\cdot(m^2-m-2)$. Hence it cannot be isometrically immersed into this codimensional Euclidean space even locally (cf. [17], [11]). Note that there still remains a gap of linear order on m between these two codimensions.)

(2) In general, the image of a polynomial map between two vector spaces is not closed, as stated before. And this phenomenon sometimes implies the existence of "almost solutions" of a system of polynomial equations. A similar phenomenon occurs in the case of left invariant torsion free flat affine connections on homogeneous spaces (cf. [6], [7], [8]). In this case, the map we consider is

$$\gamma \,:\, \{\text{left invariant torsion free affine connection}\} \longrightarrow \{\text{curvature}\},$$

which is also quadratic. The examples stated in [6], [7], [8] imply that the image of this map is not closed for the case of the three dimensional sphere. In fact, this space admits an almost flat affine structure, though it never possess torsion free flat affine connections. Our examples of almost solutions of the Gauss equation given in this paper have the same origin as in the case of this almost flat affine structure.

(3) The norms of R and $-R$ are given by

$$\|\, R \,\|^2 = \|\, -R \,\|^2 = \sum_{i<j,k<l} (R_{ijkl})^2 = 75.$$

By modifying R (or $-R$) by second fundamental forms $\gamma_r(\alpha)$, we can decrease these values. Numerical calculations by the software Mathematica, combined with the results (2) in Theorem, we have

$$\begin{array}{ll} \inf_\alpha \| \gamma_1(\alpha) - R \|^2 \leq 52.45, & \inf_\alpha \| \gamma_1(\alpha) + R \|^2 \leq 57.01, \\ \inf_\alpha \| \gamma_2(\alpha) - R \|^2 \leq 36.01, & \inf_\alpha \| \gamma_2(\alpha) + R \|^2 \leq 39.04, \\ \inf_\alpha \| \gamma_3(\alpha) - R \|^2 \leq 13.49, & \inf_\alpha \| \gamma_3(\alpha) + R \|^2 \leq 15.92, \\ \inf_\alpha \| \gamma_4(\alpha) - R \|^2 = 0, & \inf_\alpha \| \gamma_4(\alpha) + R \|^2 = 0, \end{array}$$

where α runs all over the space $S^2\mathfrak{m}^* \otimes \mathbf{R}^r$ for each r. Precise infimums for the cases $r = 1,\ 2,\ 3$ are not yet determined. But these results suggest that the least codimension where the Gauss equation admits a solution is 4 or 5 for $SU(3)/SO(3)$ and its dual.

By calculating the rank of the differential of γ_r at a generic point α, we know that there exists an obstruction to the existence of the solutions of the Gauss equation for the case $M^5 \subset \mathbf{R}^8$, which is a covariant of the curvature tensor (cf. [9]). Unfortunately, we do not know its explicit form yet. If the explicit form of the covariant is obtained, it may be possible to show that $\pm R \notin \mathrm{Im}\ \gamma_3$ by using this covariant. But, in the 4-codimensional case, it seems quite difficult to show the "non-existence" of solutions of the Gauss equation, unless we discover essentially new devices.

(4) It is an important problem whether the 5-codimensional solutions which we construct in this paper are the actual second fundamental forms for some local isometric imbeddings of $SU(3)/SO(3)$ (or its dual) into $\mathbf{R}^{10}$. It is desirable to investigate this problem by the similar method as in the case of $P^2(\mathbf{C}) \subset \mathbf{R}^7$ treated in Kaneda [16], though it requires tremendous calculations on prolongations.

(5) We consider $\| \gamma(\alpha) \pm R \|^2$ as functions on the space $S^2\mathfrak{m}^* \otimes \mathbf{R}^r$. Then, it is an interesting problem to find the critical points of these functions. Certainly, $\mathbf{R}^r$-valued symmetric forms corresponding to these critical points have some intrinsic geometric meaning, though the explicit determination of critical points seems to be quite difficult.

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