

RIGIDITY OF THE CANONICAL ISOMETRIC IMBEDDING OF THE CAYLEY PROJECTIVE PLANE $P^2(\mathbf{Cay})$

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ABSTRACT. In [7], we have proved that $P^2(\mathbf{Cay})$ cannot be isometrically immersed into \mathbf{R}^{25} even locally. In this paper, we investigate isometric immersions of $P^2(\mathbf{Cay})$ into \mathbf{R}^{26} and prove that the canonical isometric imbedding \mathbf{f}_0 of $P^2(\mathbf{Cay})$ into \mathbf{R}^{26} , which is defined in Kobayashi [17], is rigid in the following strongest sense: Any isometric immersion \mathbf{f}_1 of a connected open set $U(\subset P^2(\mathbf{Cay}))$ into \mathbf{R}^{26} coincides with \mathbf{f}_0 up to a euclidean transformation of \mathbf{R}^{26} , i.e., there is a euclidean transformation a of \mathbf{R}^{26} satisfying $\mathbf{f}_1 = a\mathbf{f}_0$ on U .

1. INTRODUCTION

In the previous paper [7], we investigated the problem of (local) isometric immersions of the quaternion projective plane $P^2(\mathbf{H})$ and the Cayley projective plane $P^2(\mathbf{Cay})$. In particular, we proved the following non-existence theorem of (local) isometric immersions:

Theorem 1. *Any open set of the Cayley projective plane $P^2(\mathbf{Cay})$ cannot be isometrically immersed into \mathbf{R}^{25} .*

As is well-known, there is an isometric immersion \mathbf{f}_0 of $P^2(\mathbf{Cay})$ into the euclidean space \mathbf{R}^{26} , which is called the canonical isometric imbedding of $P^2(\mathbf{Cay})$ (Kobayashi [17]). This fact together with Theorem 1 implies that \mathbf{R}^{26} is the least dimensional euclidean space into which $P^2(\mathbf{Cay})$ can be (locally) isometrically immersed.

In this paper, we consider (local) isometric immersions of $P^2(\mathbf{Cay})$ into \mathbf{R}^{26} and discuss the rigidity of the canonical isometric imbedding \mathbf{f}_0 . Concerning the rigidity of \mathbf{f}_0 Kaneda [15] has shown that the canonical isometric imbedding \mathbf{f}_0 is of finite type, i.e., the space of local infinitesimal isometric deformations of \mathbf{f}_0 is of finite dimension. However, it seems to the authors that any further result concerning the rigidity of \mathbf{f}_0 has not been obtained.

In the present paper, we will show the rigidity of the canonical isometric imbedding \mathbf{f}_0 in the following strongest form:

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Theorem 2. *Let \mathbf{f}_0 be the canonical isometric imbedding of $P^2(\mathbf{Cay})$ into the euclidean space \mathbf{R}^{26} . Then, for any isometric immersion \mathbf{f}_1 defined on a connected open set U of $P^2(\mathbf{Cay})$ into \mathbf{R}^{26} , there exists a euclidean transformation a of \mathbf{R}^{26} satisfying $\mathbf{f}_1 = a\mathbf{f}_0$ on U .*

To prove Theorem 2, we first establish a rigidity theorem for an isometric immersion of a Riemannian manifold. Let M be an n -dimensional Riemannian manifold and let \mathbf{f}_0 be an isometric immersion of M into the m -dimensional euclidean space \mathbf{R}^m . We will prove that if the Gauss equation in codimension $r (= m - n)$ admits essentially one solution everywhere on M , then \mathbf{f}_0 is rigid, i.e., for any isometric immersion \mathbf{f}_1 of M into \mathbf{R}^m there exists a euclidean transformation a of \mathbf{R}^m such that $\mathbf{f}_1 = a\mathbf{f}_0$ (see Theorem 5). This theorem may be established by various methods; for example, by combining the results of Nomizu [19] and Szczarba [21], [22] (cf. Agaoka [1]) or by solving a differential system of Pfaff (cf. Bishop–Crittenden [10], Ch. X). In this paper, we will give a simple proof based on a congruence theorem of differentiable mappings, which is easy to understand and gives a clear view on the geometric meaning (see Theorem 6).

Next, we will show that for the Cayley projective plane $P^2(\mathbf{Cay})$ the Gauss equation in codimension 10 ($= 26 - \dim P^2(\mathbf{Cay})$) admits essentially one solution (see Theorem 10). To show this, we utilize the results obtained in [6] and [7]. Among all, the result concerning pseudo-abelian subspaces (Proposition 8) plays an important role in our proof.

Then, Theorem 2 is a direct consequence of Theorem 5 and Theorem 10.

Throughout this paper we assume the differentiability of class C^∞ . Notations for Lie algebras are the same that are used in [6] and [7].

2. THE GAUSS EQUATION

Let M be a Riemannian manifold and $T(M)$ the tangent bundle of M . We denote by g the Riemannian metric of M and by R the Riemannian curvature tensor of type $(1, 3)$ with respect to g .

Let \mathbf{N} be a euclidean vector space, i.e., \mathbf{N} is a vector space over \mathbf{R} endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $p \in M$ and let $S^2T_p^*(M) \otimes \mathbf{N}$ be the space of \mathbf{N} -valued symmetric bilinear forms on $T_p(M)$. We call the following equation on $\Psi \in S^2T_p^*(M) \otimes \mathbf{N}$ the *Gauss equation* at $p \in M$:

$$-g_p(R_p(x, y)z, w) = \langle \Psi(x, z), \Psi(y, w) \rangle - \langle \Psi(x, w), \Psi(y, z) \rangle, \quad (2.1)$$

where $x, y, z, w \in T_p(M)$. We denote by $\mathcal{G}_p(\mathbf{N})$ the set of all solutions of (2.1), which is called the *Gaussian variety* associated with \mathbf{N} at $p \in M$. As is well-known, $\mathcal{G}_p(\mathbf{N}) = \emptyset$

happens in case the dimensionality $r (= \dim \mathbf{N})$ is so small, however, $\mathcal{G}_p(\mathbf{N}) \neq \emptyset$ if r is sufficiently large (see Cartan [11] or Kaneda–Tanaka [16]).

Let \mathbf{N}_1 and \mathbf{N}_2 be two euclidean vector spaces and let φ be a linear mapping of \mathbf{N}_1 to \mathbf{N}_2 . Define a linear map $\widehat{\varphi}$ of $S^2T_p^*(M) \otimes \mathbf{N}_1$ to $S^2T_p^*(M) \otimes \mathbf{N}_2$ by

$$(\widehat{\varphi} \Psi)(x, y) = \varphi(\Psi(x, y)), \quad \Psi \in S^2T_p^*(M) \otimes \mathbf{N}_1, \quad x, y \in T_p(M). \quad (2.2)$$

Then, we can easily verify

Lemma 3. *Let φ be a linear mapping of a euclidean vector space \mathbf{N}_1 to a euclidean vector space \mathbf{N}_2 . Assume that φ is isometric, i.e., $\langle \varphi(x), \varphi(y) \rangle_2 = \langle x, y \rangle_1$ ($x, y \in T_p(M)$), where $\langle \cdot, \cdot \rangle_i$ ($i = 1, 2$) denotes the inner product of \mathbf{N}_i . Then $\widehat{\varphi} \mathcal{G}_p(\mathbf{N}_1) \subset \mathcal{G}_p(\mathbf{N}_2)$. In particular, if $\dim \mathbf{N}_1 = \dim \mathbf{N}_2$, then $\widehat{\varphi} \mathcal{G}_p(\mathbf{N}_1) = \mathcal{G}_p(\mathbf{N}_2)$.*

In view of Lemma 3, the solvability of the Gauss equation (2.1) substantially depends on the dimensionality of \mathbf{N} . To emphasize $\dim \mathbf{N}$ we call (2.1) the Gauss equation in codimension $r (= \dim \mathbf{N})$.

Let \mathbf{N} be a euclidean vector space and let $O(\mathbf{N})$ be the orthogonal transformation group of \mathbf{N} . We define an action of $O(\mathbf{N})$ on $S^2T_p^*(M) \otimes \mathbf{N}$ by

$$(h\Psi)(x, y) = h(\Psi(x, y)),$$

where $\Psi \in S^2T_p^*(M) \otimes \mathbf{N}$, $h \in O(\mathbf{N})$, $x, y \in T_p(M)$. We say that two elements Ψ and $\Psi' \in S^2T_p^*(M) \otimes \mathbf{N}$ are *equivalent* if there is an element $h \in O(\mathbf{N})$ such that $\Psi' = h\Psi$. It is easily seen that if Ψ and $\Psi' \in S^2T_p^*(M) \otimes \mathbf{N}$ are equivalent and $\Psi \in \mathcal{G}_p(\mathbf{N})$, then $\Psi' \in \mathcal{G}_p(\mathbf{N})$. We say that the Gaussian variety $\mathcal{G}_p(\mathbf{N})$ is *EOS* if $\mathcal{G}_p(\mathbf{N}) \neq \emptyset$ and if it is composed of essentially one solution, i.e., any solutions of the Gauss equation (2.1) are equivalent to each other under the action of $O(\mathbf{N})$.

Proposition 4. *Let M be a Riemannian manifold and let $p \in M$. Let \mathbf{N} be an r -dimensional euclidean vector space such that $\mathcal{G}_p(\mathbf{N})$ is EOS. Then:*

- (1) *Let Ψ be an arbitrary element of $\mathcal{G}_p(\mathbf{N})$. Then, the vectors $\Psi(x, y)$ ($x, y \in T_p(M)$) span the whole space \mathbf{N} .*
- (2) *Let \mathbf{N}_1 be a euclidean vector space. Then:*
 - (2a) $\mathcal{G}_p(\mathbf{N}_1) = \emptyset$ if $\dim \mathbf{N}_1 < r$;
 - (2b) $\mathcal{G}_p(\mathbf{N}_1)$ is EOS if $\dim \mathbf{N}_1 = r$;
 - (2c) $\mathcal{G}_p(\mathbf{N}_1)$ is not EOS if $\dim \mathbf{N}_1 > r$.

Proof. Note that if $\Psi' \in S^2T_p^*(M) \otimes \mathbf{N}$ is equivalent to Ψ , then we have $|\Psi'(x, y)| = |\Psi(x, y)|$ for any $x, y \in T_p(M)$, where $|\mathbf{n}|$ denotes the norm of $\mathbf{n} \in \mathbf{N}$ with respect to $\langle \cdot, \cdot \rangle$.

Now, suppose that the vectors $\Psi(x, y)$ ($x, y \in T_p(M)$) do not span the whole space \mathbf{N} . Then, there is a non-zero vector $\mathbf{n} \in \mathbf{N}$ satisfying $\langle \mathbf{n}, \Psi(x, y) \rangle = 0$ for any $x, y \in T_p(M)$. Define an element $\Psi' \in S^2T_p^*(M) \otimes \mathbf{N}$ by

$$\Psi' = \Psi + (\xi^*)^2 \otimes \mathbf{n},$$

where ξ^* is a non-zero element of $T_p^*(M)$. Then, it is easy to see that $\Psi' \in \mathcal{G}_p(\mathbf{N})$. However, by a simple calculation, we have $|\Psi'(x, x)|^2 = |\Psi(x, x)|^2 + |\mathbf{n}|^2 \xi^*(x)^2$. Therefore, if we take $x \in T_p(M)$ such that $\xi^*(x) \neq 0$, then we have $|\Psi'(x, x)| \neq |\Psi(x, x)|$. This proves that Ψ' is not equivalent to Ψ and hence $\mathcal{G}_p(\mathbf{N})$ is not EOS. Thus, we obtain (1).

Next we prove (2). First assume $\dim \mathbf{N}_1 = r$. Let φ be an isometric linear isomorphism of \mathbf{N} onto \mathbf{N}_1 . Then we have $O(\mathbf{N}_1) = \varphi \cdot O(\mathbf{N}) \cdot \varphi^{-1}$. Moreover, by Lemma 3 we have $\widehat{\varphi} \mathcal{G}_p(\mathbf{N}) = \mathcal{G}_p(\mathbf{N}_1)$. Since $\mathcal{G}_p(\mathbf{N})$ is EOS, $O(\mathbf{N})$ acts transitively on $\mathcal{G}_p(\mathbf{N})$. Therefore, it is easily seen that $O(\mathbf{N}_1)$ acts transitively on $\mathcal{G}_p(\mathbf{N}_1)$. This proves that $\mathcal{G}_p(\mathbf{N}_1)$ is EOS.

We next consider the case $\dim \mathbf{N}_1 < r$. Suppose that $\mathcal{G}_p(\mathbf{N}_1) \neq \emptyset$ and $\Psi_1 \in \mathcal{G}_p(\mathbf{N}_1)$. Let φ be an isometric linear mapping of \mathbf{N}_1 to \mathbf{N} . Then, we know that $\widehat{\varphi} \Psi_1 \in \mathcal{G}_p(\mathbf{N})$ and the vectors $(\widehat{\varphi} \Psi_1)(x, y)$ ($x, y \in T_p(M)$) are contained in the proper subspace $\varphi(\mathbf{N}_1)$ ($\subsetneq \mathbf{N}$). This contradicts (1). The case $\dim \mathbf{N}_1 > r$ is similarly dealt with. \square

We say that a Riemannian manifold M is *formally rigid* in codimension r if there is a euclidean vector space \mathbf{N} with $\dim \mathbf{N} = r$ such that the Gaussian variety $\mathcal{G}_p(\mathbf{N})$ is EOS at each $p \in M$. By virtue of Proposition 4 (2), we know that if M is formally rigid in codimension r , then it is not formally rigid in any other codimension $r' (\neq r)$.

Remark 1. It should be noted that there is a Riemannian manifold M that is not formally rigid in any codimension r . For example, assume that M is the space of negative constant curvature of dimension n . Let \mathbf{N} be a euclidean vector space of dimension r . Then, by Ôtsuki's lemma we have $\mathcal{G}_p(\mathbf{N}) = \emptyset$ if $r < n - 1$ (see Ôtsuki [20]). On the other hand, Kaneda [13] proved that if $r = n - 1$, then $\mathcal{G}_p(\mathbf{N}) \neq \emptyset$ and around a suitable $\Psi_0 \in \mathcal{G}_p(\mathbf{N})$, $\mathcal{G}_p(\mathbf{N})$ forms a submanifold of $S^2T_p^*(M) \otimes \mathbf{N}$ of dimension $n(n-1)$ (see Theorem 3.1 of [13]). Since $n(n-1) > \dim O(\mathbf{N})$, $\mathcal{G}_p(\mathbf{N})$ cannot be EOS. If $r \geq n$, then by Proposition 4 (2a) we know that $\mathcal{G}_p(\mathbf{N})$ is not EOS. Accordingly, the space of negative constant curvature M is not formally rigid in any codimension r .

Remark 2. For each Riemannian submanifold $M \subset \mathbf{R}^m$ listed below, $\mathcal{G}_p(\mathbf{N})$ is known to be EOS at each $p \in M$, where \mathbf{N} is the normal vector space of M at p in \mathbf{R}^m :

- (1) The sphere $S^n \subset \mathbf{R}^{n+1}$ ($n \geq 3$);
- (2) The symplectic group $Sp(2) \subset \mathbf{R}^{16}$ (see Agaoka [1]);
- (3) A submanifold $M \subset \mathbf{R}^m$ with type number ≥ 3 (see Allendoerfer [9], Kobayashi-Nomizu [18]).

Consequently, these submanifolds are formally rigid in our sense and it has been proved that they are actually rigid in \mathbf{R}^m (see [1], [9]).

However, we note that the formal rigidness of M in codimension r does not imply the existence of an isometric immersion of M into \mathbf{R}^{n+r} ($n = \dim M$). Indeed, Kaneda [14] gave an example of three dimensional Riemannian manifold M that is formally rigid in codimension 1 but cannot be locally isometrically immersed into \mathbf{R}^4 .

We will prove in the next section that if a connected Riemannian manifold M is formally rigid in codimension r and if there is an isometric immersion \mathbf{f} of M into \mathbf{R}^{n+r} ($n = \dim M$), then M (precisely, $\mathbf{f}(M)$) is actually rigid in \mathbf{R}^{n+r} (see Theorem 5).

3. RIGIDITY THEOREM

In this section, we will prove the following rigidity theorem:

Theorem 5. *Let M be an n -dimensional Riemannian manifold and let \mathbf{f}_0 be an isometric immersion of M into the euclidean space \mathbf{R}^m . Assume:*

- (1) M is connected;
- (2) M is formally rigid in codimension $r = m - n$.

Then, any isometric immersion \mathbf{f}_1 of M into the euclidean space \mathbf{R}^m coincides with \mathbf{f}_0 up to a euclidean transformation of \mathbf{R}^m , i.e., there exists a euclidean transformation a of \mathbf{R}^m such that $\mathbf{f}_1 = a\mathbf{f}_0$.

Before proceeding to the proof of Theorem 5, we make some preparations. Let $M(m, m')$ be the space of real matrices of degree $m \times m'$, where m and m' are non-negative integers. In what follows we identify $M(m, 1)$ with the m -dimensional euclidean space \mathbf{R}^m in a natural way. Then, we note that the canonical inner product $\langle \cdot, \cdot \rangle$ of \mathbf{R}^m is given by $\langle \mathbf{v}, \mathbf{w} \rangle = {}^t\mathbf{v} \cdot \mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in \mathbf{R}^m$.

Let us define an operation of $M(m, m)$ on \mathbf{R}^m by

$$M(m, m) \times \mathbf{R}^m \ni (H, \mathbf{v}) \longmapsto H \cdot \mathbf{v} \in \mathbf{R}^m,$$

where \cdot means the usual matrix multiplication.

Let ∇ be the Riemannian connection associated with M . Let $\mathbf{f} = {}^t(f^1, \dots, f^m)$ be a differentiable map of M into the euclidean space \mathbf{R}^m . By $\overbrace{\nabla \cdots \nabla}^k \mathbf{f}$ we denote the k -th order covariant derivative of \mathbf{f} , which is defined as follows:

$$\overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^k \mathbf{f} = {}^t(\dots, \overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^k f^i, \dots) \in \mathbf{R}^m,$$

where $p \in M$; $x_1, \dots, x_k \in T_p(M)$. (Precisely, see Tanaka [23], Kaneda–Tanaka [16] or Kaneda [14].) It is known that $\nabla\nabla\mathbf{f}$ and $\nabla\nabla\nabla\mathbf{f}$ satisfy the following integrability conditions:

$$\nabla_x\nabla_y\mathbf{f} = \nabla_y\nabla_x\mathbf{f}, \quad (3.1)$$

$$\nabla_z\nabla_x\nabla_y\mathbf{f} = \nabla_x\nabla_z\nabla_y\mathbf{f} - \nabla_{R(z,x)y}\mathbf{f}. \quad (3.2)$$

We say that a differentiable map \mathbf{f} of M into \mathbf{R}^m is *2-generic* if at each $p \in M$, the whole space \mathbf{R}^m is spanned by the vectors of the form $\nabla_x\mathbf{f}$ ($x \in T_p(M)$), $\nabla_y\nabla_z\mathbf{f}$ ($y, z \in T_p(M)$). It is clear that if \mathbf{f} is 2-generic, then we have the inequality $m \leq \frac{1}{2}n(n+3)$. Note that a 2-generic map \mathbf{f} is not necessarily an immersion.

We first show the following congruence theorem:

Theorem 6. *Let M be an n -dimensional Riemannian manifold and let \mathbf{f}_i ($i = 0, 1$) be two differentiable maps of M into the euclidean space \mathbf{R}^m . Assume:*

- (1) M is connected;
- (2) \mathbf{f}_0 is 2-generic;
- (3) At each $p \in M$ there is an element $H(p) \in O(m)$ satisfying

$$\nabla_x\mathbf{f}_1 = H(p) \cdot (\nabla_x\mathbf{f}_0) \quad \forall x \in T_p(M), \quad (3.3)$$

$$\nabla_y\nabla_z\mathbf{f}_1 = H(p) \cdot (\nabla_y\nabla_z\mathbf{f}_0), \quad \forall y, z \in T_p(M). \quad (3.4)$$

Then, \mathbf{f}_1 coincides with \mathbf{f}_0 up to a euclidean transformation of \mathbf{R}^m . More precisely, $H(p)$ is identically equal to a constant value $H_0 \in O(m)$ everywhere on M and \mathbf{f}_1 can be written as $\mathbf{f}_1 = H_0\mathbf{f}_0 + \mathbf{c}_0$, where \mathbf{c}_0 is a constant vector of \mathbf{R}^m .

Proof. We first note that, since \mathbf{f}_0 is 2-generic, $H(p)$ satisfying (3.3) and (3.4) is uniquely determined at each $p \in M$ and the map $H: M \ni p \mapsto H(p) \in O(m)$ is differentiable. Via the canonical inclusion $O(m) \subset M(m, m)$, we can regard H as an $M(m, m)$ -valued function on M satisfying

$${}^tHH = I_m, \quad (3.5)$$

where I_m denotes the identity matrix of degree m . Differentiate (3.5) covariantly. Then by Leibnitz' law we get

$$\nabla_x({}^tH)H(p) + {}^tH(p)(\nabla_xH) = 0, \quad \forall x \in T_p(M). \quad (3.6)$$

In this equality, the covariant derivative ∇_xH means the element of $M(m, m)$ given by $\nabla_xH = (\nabla_xh_i^j)$, where h_i^j denotes the (i, j) -component of H . By the very definition of ∇_xH we have $\nabla_x({}^tH) = {}^t(\nabla_xH)$.

Let us define an $M(m, m)$ -valued 1-form L by

$$L(x) = {}^tH(p)(\nabla_x H), \quad x \in T_p(M). \quad (3.7)$$

Then, by (3.6) we have

$${}^tL(x) + L(x) = 0, \quad \forall x \in T_p(M), \quad (3.8)$$

implying that the matrix $L(x) \in M(m, m)$ is skew-symmetric.

We now show that the equality $L(x) = 0$ holds for any $x \in T_p(M)$. Since \mathbf{f}_0 is 2-generic, it suffices to prove

$$L(y) \cdot (\nabla_x \mathbf{f}_0) = 0, \quad \forall x, y \in T_p(M), \quad (3.9)$$

$$L(z) \cdot (\nabla_y \nabla_x \mathbf{f}_0) = 0, \quad \forall x, y, z \in T_p(M). \quad (3.10)$$

Differentiating (3.3) and (3.4) covariantly, we have

$$\nabla_y \nabla_x \mathbf{f}_1 = \nabla_y H \cdot (\nabla_x \mathbf{f}_0) + H(p) \cdot (\nabla_y \nabla_x \mathbf{f}_0), \quad \forall x, y \in T_p(M), \quad (3.11)$$

$$\nabla_z \nabla_y \nabla_x \mathbf{f}_1 = \nabla_z H \cdot (\nabla_y \nabla_x \mathbf{f}_0) + H(p) \cdot (\nabla_z \nabla_y \nabla_x \mathbf{f}_0), \quad \forall x, y, z \in T_p(M). \quad (3.12)$$

Then by (3.4) and (3.11) we have $\nabla_y H \cdot (\nabla_x \mathbf{f}_0) = 0$ for each $x, y \in T_p(M)$. Consequently, multiplying ${}^tH(p)$ from the left, we have (3.9).

We now prove (3.10). Exchanging z and y in (3.12), we have

$$\nabla_y \nabla_z \nabla_x \mathbf{f}_1 = \nabla_y H \cdot (\nabla_z \nabla_x \mathbf{f}_0) + H(p) \cdot (\nabla_y \nabla_z \nabla_x \mathbf{f}_0), \quad \forall x, y, z \in T_p(M). \quad (3.13)$$

Subtract (3.13) from (3.12). Then, using the integrability condition (3.2) and the equality (3.3), we have

$$\nabla_z H(\nabla_y \nabla_x \mathbf{f}_0) = \nabla_y H(\nabla_z \nabla_x \mathbf{f}_0), \quad \forall x, y, z \in T_p(M). \quad (3.14)$$

Consequently, multiplying ${}^tH(p)$ from the left, we get

$$L(z) \cdot (\nabla_y \nabla_x \mathbf{f}_0) = L(y) \cdot (\nabla_z \nabla_x \mathbf{f}_0), \quad \forall x, y, z \in T_p(M). \quad (3.15)$$

Since $L(z)$ is a skew-symmetric matrix, we have

$$\langle L(z) \cdot (\nabla_y \nabla_x \mathbf{f}_0), \nabla_u \mathbf{f}_0 \rangle = -\langle (\nabla_y \nabla_x \mathbf{f}_0), L(z) \cdot (\nabla_u \mathbf{f}_0) \rangle = 0.$$

Therefore, to prove (3.10), we have to show

$$\langle L(z) \cdot (\nabla_y \nabla_x \mathbf{f}_0), \nabla_v \nabla_w \mathbf{f}_0 \rangle = 0, \quad \forall x, y, z, v, w \in T_p(M). \quad (3.16)$$

Define an element $X \in \otimes^5 T_p^*(M)$ by

$$X(z, y, x, v, w) = \langle L(z) \cdot (\nabla_y \nabla_x \mathbf{f}_0), \nabla_v \nabla_w \mathbf{f}_0 \rangle, \quad x, y, z, v, w \in T_p(M). \quad (3.17)$$

In the following, we will show $X(z, y, x, v, w) = 0$ for $x, y, z, v, w \in T_p(M)$. By the integrability condition (3.1) and by (3.15), we easily know that $X(z, y, x, v, w)$ is symmetric with

respect to the pairs $\{x, y\}$, $\{v, w\}$ and $\{z, y\}$. Further, since $L(z)$ is a skew-symmetric endomorphism of \mathbf{R}^m (see (3.8)), it follows that

$$X(z, y, x, v, w) = -X(z, v, w, y, x). \quad (3.18)$$

Therefore, $X(z, y, x, v, w)$ is anti-symmetric with respect to the pair $\{x, w\}$, because

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, v, w, y, x) = -X(v, z, w, y, x) \\ &= X(v, y, x, z, w) = X(y, v, x, z, w) \\ &= -X(y, z, w, v, x) = -X(z, y, w, v, x). \end{aligned}$$

Consequently, we get

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, y, w, v, x) = -X(z, w, y, x, v) \\ &= X(z, w, v, x, y) = X(z, v, w, y, x). \end{aligned}$$

This together with (3.18) proves $X(z, y, x, v, w) = 0$. Thus we get (3.10).

By the above argument, we know that $L(x) = {}^tH(p)(\nabla_x H) = 0$ for any $x \in T_p(M)$. This implies that H is a locally constant function and hence H is identically equal to an element $H_0 \in O(m)$ on M , because M is connected. Consequently, the difference $\mathbf{c} = \mathbf{f}_1 - H_0 \cdot \mathbf{f}_0$ satisfies

$$\nabla_x \mathbf{c} = \nabla_x (\mathbf{f}_1 - H_0 \cdot \mathbf{f}_0) = \nabla_x \mathbf{f}_1 - H_0 \cdot (\nabla_x \mathbf{f}_0) = 0, \quad \forall x \in T_p(M).$$

Therefore, \mathbf{c} is also identically equal to a constant vector $\mathbf{c}_0 \in \mathbf{R}^m$, completing the proof of the theorem. \square

Remark 3. The argument in the proof of the equality $X = 0$ is essentially the same that is developed in the proof of the uniqueness of the metric connection of the normal bundle associated with an isometric imbedding (see the proof of Theorem 1 of [19]); It is almost the same that is used to calculate the third prolongation of the symbol of the operator L (see Proposition 2.2 of [16]). Here we remark that $X = 0$ can be proved without assuming the existence of (isometric) immersions.

We are now in a position to prove Theorem 5.

Proof of Theorem 5. We show that the map \mathbf{f}_i ($i = 0, 1$) is 2-generic and for each $p \in M$ there is an element $H(p) \in O(m)$ satisfying the equalities (3.3) and (3.4).

Let $i = 0$ or 1. Let $\mathbf{f}_{i*}T_p(M)$ (resp. \mathbf{N}_i) be the tangent vector space (resp. normal vector space) of $\mathbf{f}_i(M)$ at $\mathbf{f}_i(p) \in \mathbf{R}^m$. Then, we have $\dim \mathbf{f}_{i*}T_p(M) = n$ and $\dim \mathbf{N}_i = m - n$. We regard $\mathbf{f}_{i*}T_p(M)$ and \mathbf{N}_i as euclidean vector spaces endowed with the inner products induced from the inner product $\langle \cdot, \cdot \rangle$ of \mathbf{R}^m . By a natural parallel displacement from $\mathbf{f}_i(p)$

to the origin $o \in \mathbf{R}^m$, we regard $\mathbf{f}_{i*}T_p(M)$ and \mathbf{N}_i as linear subspaces of \mathbf{R}^m . Since \mathbf{f}_i is an isometric immersion, $\mathbf{f}_{i*}T_p(M)$ is spanned by the vectors $\nabla_x \mathbf{f}_i$ ($x \in T_p(M)$) and

$$\langle \nabla_x \mathbf{f}_i, \nabla_y \mathbf{f}_i \rangle = g_p(x, y), \quad x, y \in T_p(M). \quad (3.19)$$

The second order derivative $\nabla \nabla \mathbf{f}_i$, which is so called the *second fundamental form* of \mathbf{f}_i , satisfies $\nabla \nabla \mathbf{f}_i \in S^2 T_p^*(M) \otimes \mathbf{N}_i$ and $\nabla \nabla \mathbf{f}_i \in \mathcal{G}_p(\mathbf{N}_i)$ (see [23], [16]). Since $\mathcal{G}_p(\mathbf{N}_i)$ is EOS, the vectors $\nabla_x \nabla_y \mathbf{f}_i$ ($x, y \in T_p(M)$) span \mathbf{N}_i , implying that \mathbf{f}_i is 2-generic (see Proposition 4 (1)). Take an isometric linear isomorphism φ_2 of \mathbf{N}_0 onto \mathbf{N}_1 . Since $\widehat{\varphi}_2 \nabla \nabla \mathbf{f}_0 \in \mathcal{G}_p(\mathbf{N}_1)$ and since $\mathcal{G}_p(\mathbf{N}_1)$ is EOS (see Proposition 4 (2b)), there is an element $h_1 \in O(\mathbf{N}_1)$ such that $h_1(\widehat{\varphi}_2 \nabla \nabla \mathbf{f}_0) = \nabla \nabla \mathbf{f}_1$. On the other hand, in view of (3.19) we also know that there is an isometric linear isomorphism φ_1 of $\mathbf{f}_{0*}T_p(M)$ onto $\mathbf{f}_{1*}T_p(M)$ satisfying $\varphi_1(\nabla_x \mathbf{f}_0) = \nabla_x \mathbf{f}_1$ ($x \in T_p(M)$). Define a linear endomorphism $H(p)$ of \mathbf{R}^m satisfying $H(p)|_{\mathbf{f}_{0*}T_p(M)} = \varphi_1$ and $H(p)|_{\mathbf{N}_0} = h_1 \cdot \varphi_2$. Then, it is easily seen that $H(p) \in O(m)$ and the equalities (3.3) and (3.4) are satisfied.

Therefore, by Theorem 6 we know that \mathbf{f}_1 can be written as $\mathbf{f}_1 = a\mathbf{f}_0$, where a denotes the euclidean transformation of \mathbf{R}^m defined by $\mathbf{R}^m \ni \mathbf{x} \mapsto H_0 \cdot \mathbf{x} + \mathbf{c}_0 \in \mathbf{R}^m$. Thus, we obtain the theorem. \square

4. THE CAYLEY PROJECTIVE PLANE $P^2(\mathbf{Cay})$

Let $M = G/K$ be a compact Riemannian symmetric space. Let \mathfrak{g} (resp. \mathfrak{k}) be the Lie algebra of G (resp. K). We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the canonical decomposition of \mathfrak{g} associated with the symmetric pair (G, K) . We denote by (\cdot, \cdot) the inner product of \mathfrak{g} given by the (-1) -multiple of the Killing form of \mathfrak{g} . As usual, we can identify \mathfrak{m} with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G -invariant Riemannian metric g of G/K satisfies

$$g_o(X, Y) = (X, Y), \quad X, Y \in \mathfrak{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type (1, 3) is given by

$$R_o(X, Y)Z = -[[X, Y], Z], \quad \forall X, Y, Z \in \mathfrak{m}.$$

Hereafter, we consider the case of the Cayley projective plane $P^2(\mathbf{Cay})$. As is well-known, $P^2(\mathbf{Cay})$ can be represented by $P^2(\mathbf{Cay}) = G/K$, where $G = F_4$ and $K = Spin(9)$. Take a maximal abelian subspace \mathfrak{a} of \mathfrak{m} and fix it in the following discussions. We note that since $\text{rank}(P^2(\mathbf{Cay})) = 1$, we have $\dim \mathfrak{a} = 1$.

For each element $\lambda \in \mathfrak{a}$ we define two subspaces $\mathfrak{k}(\lambda) \subset \mathfrak{k}$ and $\mathfrak{m}(\lambda) \subset \mathfrak{m}$ by

$$\mathfrak{k}(\lambda) = \left\{ X \in \mathfrak{k} \mid [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a} \right\},$$

$$\mathfrak{m}(\lambda) = \left\{ Y \in \mathfrak{m} \mid [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a} \right\}.$$

We call λ a *restricted root* if $\mathfrak{m}(\lambda) \neq 0$. Let Σ be the set of all non-zero restricted roots. In the case of $P^2(\mathbf{Cay})$, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ . Then we have $\mathfrak{m}(0) = \mathfrak{a} = \mathbf{R}\mu$ and

$$\begin{aligned} \mathfrak{k} &= \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \text{ (orthogonal direct sum),} \\ \mathfrak{m} &= \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \text{ (orthogonal direct sum).} \end{aligned}$$

(For details, see [6], [7].) For simplicity, for each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ ($|i| \leq 2$), $\mathfrak{k}_i = \mathfrak{m}_i = 0$ ($|i| > 2$). Then we have

Proposition 7 ([7]). (1) *Let $i, j = 0, 1, 2$. Then:*

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \quad (4.1)$$

(2) $\dim \mathfrak{m} = 16$, $\dim \mathfrak{k}_1 = \dim \mathfrak{m}_1 = 8$, $\dim \mathfrak{k}_2 = \dim \mathfrak{m}_2 = 7$.

In what follows, we recall the results obtained in [7], which will be needed in the proof of Theorem 2. Let V be a subspace of \mathfrak{m} . V is called *pseudo-abelian* if it satisfies $[V, V] \subset \mathfrak{k}_0$ (or equivalently $[[V, V], \mathfrak{a}] = 0$). (Precisely, see [6].) As is easily seen, \mathfrak{m}_2 is a pseudo-abelian subspace of \mathfrak{m} , because $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}_0$ (see (4.1)).

On the contrary, we have

Proposition 8. *Let $G/K = P^2(\mathbf{Cay})$. Then, any pseudo-abelian subspace V of \mathfrak{m} with $\dim V > 2$ must be contained in \mathfrak{m}_2 .*

For the proof, see Lemma 6 of [7]. The following proposition summarizes the results of [7] (see Proposition 7, Proposition 10 and Lemma 17 of [7]).

Proposition 9. (1) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Assume that $Y_0 \neq 0$, $Y_1 \neq 0$. Then, there are elements $k_0, k_1 \in K$ satisfying*

$$\text{Ad}(k_0)\mu \in \mathbf{R}Y_0, \quad \text{Ad}(k_0)\mathfrak{m}_2 = \{Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y'_0, Y_0) = 0\}, \quad (4.2)$$

$$\text{Ad}(k_1)\mu \in \mathbf{R}Y_1, \quad \text{Ad}(k_1)\mathfrak{m}_2 = \{Y'_1 \in \mathfrak{m}_1 \mid (Y'_1, Y_1) = 0\}. \quad (4.3)$$

(2) *Let $Y_0, Y'_0 \in \mathfrak{a} + \mathfrak{m}_2$, $Y_1, Y'_1 \in \mathfrak{m}_1$ and $X_1 \in \mathfrak{k}_1$. Then:*

$$[Y_0, [Y_0, Y'_0]] = \begin{cases} -4(\mu, \mu)(Y_0, Y_0)Y'_0, & \text{if } (Y_0, Y'_0) = 0, \\ 0, & \text{if } Y'_0 \in \mathbf{R}Y_0, \end{cases} \quad (4.4)$$

$$[Y_0, [Y_0, Y_1]] = -(\mu, \mu)(Y_0, Y_0)Y_1, \quad (4.5)$$

$$[Y_1, [Y_1, Y_0]] = -(\mu, \mu)(Y_1, Y_1)Y_0, \quad (4.6)$$

$$[Y_1, [Y_1, Y'_1]] = \begin{cases} -4(\mu, \mu)(Y_1, Y_1) Y'_1, & \text{if } (Y_1, Y'_1) = 0, \\ 0, & \text{if } Y'_1 \in \mathbf{R}Y_1, \end{cases} \quad (4.7)$$

$$[X_1, [X_1, Y_0]] = -(\mu, \mu)(X_1, X_1) Y_0. \quad (4.8)$$

5. SOLUTIONS OF THE GAUSS EQUATION

In this and the next sections, we prove

Theorem 10. *The projective plane $P^2(\mathbf{Cay})$ is formally rigid in codimension 10 (= 26 – dim $P^2(\mathbf{Cay})$).*

If this theorem is established, then Theorem 2 immediately follows from Theorem 5.

On account of homogeneity of $P^2(\mathbf{Cay})$, in order to show Theorem 10 we have only to prove that the Gaussian variety $\mathcal{G}_o(\mathbf{N})$ is EOS at the origin o for any euclidean vector space \mathbf{N} with $\dim \mathbf{N} = 10$.

In what follows we assume that $M = P^2(\mathbf{Cay})$ and that \mathbf{N} is a euclidean vector space with $\dim \mathbf{N} = 10$. We will prove the following theorem:

Theorem 11. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Then:*

(1) *There are linearly independent vectors \mathbf{A} and $\mathbf{B} \in \mathbf{N}$ satisfying*

$$(1a) \quad \langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu) \text{ and } \langle \mathbf{A}, \mathbf{B} \rangle = 2(\mu, \mu);$$

$$(1b) \quad \Psi(Y_0, Y'_0) = (Y_0, Y'_0) \mathbf{A}, \quad \forall Y_0, Y'_0 \in \mathfrak{a} + \mathfrak{m}_2;$$

$$(1c) \quad \Psi(Y_1, Y'_1) = (Y_1, Y'_1) \mathbf{B}, \quad \forall Y_1, Y'_1 \in \mathfrak{m}_1;$$

$$(1d) \quad \langle \mathbf{A}, \Psi(\mu, \mathfrak{m}_1) \rangle = \langle \mathbf{B}, \Psi(\mu, \mathfrak{m}_1) \rangle = 0.$$

$$(2) \quad \Psi(Y_1, Y_2) + \frac{1}{(\mu, \mu)^2} \Psi(\mu, [[\mu, Y_1], Y_2]) = 0, \quad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2.$$

$$(3) \quad \langle \Psi(\mu, Y_1), \Psi(\mu, Y'_1) \rangle = (\mu, \mu)^2 (Y_1, Y'_1), \quad \forall Y_1, Y'_1 \in \mathfrak{m}_1.$$

Before proceeding to the proof of Theorem 11 we make a somewhat lengthy preparation. Let \mathbf{N} be a euclidean vector space and let $S^2\mathfrak{m}^* \otimes \mathbf{N}$ be the space of \mathbf{N} -valued symmetric bilinear forms on \mathfrak{m} . Let $\Psi \in S^2\mathfrak{m}^* \otimes \mathbf{N}$ and $Y \in \mathfrak{m}$. We define a linear map Ψ_Y of \mathfrak{m} to \mathbf{N} by

$$\Psi_Y: \mathfrak{m} \ni Y' \mapsto \Psi(Y, Y') \in \mathbf{N},$$

and denote by $\mathbf{Ker}(\Psi_Y)$ the kernel of Ψ_Y . We say that an element $Y \in \mathfrak{m}$ is *singular* (resp. *non-singular*) with respect to Ψ if $\Psi_Y(\mathfrak{m}) \neq \mathbf{N}$ (resp. $\Psi_Y(\mathfrak{m}) = \mathbf{N}$). Apparently, $0(\in \mathfrak{m})$ is a singular element for any $\Psi \in S^2\mathfrak{m}^* \otimes \mathbf{N}$.

Proposition 12. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Let $Y \in \mathfrak{m}$ ($Y \neq 0$) and let k be an element of K satisfying $\text{Ad}(k)\mu \in \mathbf{R}Y$. Then:*

- (1) $\mathbf{Ker}(\Psi_Y) \subset \text{Ad}(k)\mathfrak{m}_2$. Consequently, $\dim \mathbf{Ker}(\Psi_Y) \leq 7$.
- (2) Assume that Y is non-singular with respect to Ψ . Then, it holds that $\dim \mathbf{Ker}(\Psi_Y) = 6$ and $\mathbf{Ker}(\Psi_Y) \subsetneq \text{Ad}(k)\mathfrak{m}_2$.
- (3) Assume that Y is singular with respect to Ψ . Then, it holds that $\mathbf{Ker}(\Psi_Y) = \text{Ad}(k)\mathfrak{m}_2$, $\dim \mathbf{Ker}(\Psi_Y) = 7$ and $\dim \Psi_Y(\mathfrak{m}) = 9$.

Proof. First, note that $\dim \mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m} - \dim \mathbf{N} = 6$. Consequently, it is easy to see that Y is singular (resp. non-singular) with respect to Ψ if and only if $\dim \mathbf{Ker}(\Psi_Y) > 6$ (resp. $\dim \mathbf{Ker}(\Psi_Y) = 6$).

Multiplying Y by a non-zero scalar if necessary, we may assume that $Y = \text{Ad}(k)\mu$. We have proved in § 1 of [3]

$$R_o(\mathbf{Ker}(\Psi_Y), \mathbf{Ker}(\Psi_Y))Y = 0.$$

In our terminology we have

$$[[\mathbf{Ker}(\Psi_Y), \mathbf{Ker}(\Psi_Y)], Y] = 0.$$

Applying $\text{Ad}(k^{-1})$ to the both sides of the above equality, we have

$$[[\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y), \text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)], \mu] = 0.$$

Since $\mathfrak{a} = \mathbf{R}\mu$, it follows that $\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)$ is a pseudo-abelian subspace of \mathfrak{m} . By Proposition 8 and by the fact $\dim \mathbf{Ker}(\Psi_Y) \geq 6$, we have $\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) \subset \mathfrak{m}_2$ and hence $\mathbf{Ker}(\Psi_Y) \subset \text{Ad}(k)\mathfrak{m}_2$, proving (1).

Assume that Y is non-singular with respect to Ψ . Then, as we have stated above, we have $\dim \mathbf{Ker}(\Psi_Y) = 6$. Since $\dim \mathfrak{m}_2 = 7$ (see Proposition 7 (2)), it follows that $\mathbf{Ker}(\Psi_Y) \subsetneq \text{Ad}(k)\mathfrak{m}_2$, proving (2).

Finally, we assume Y is singular with respect to Ψ . Then, we have $\dim \mathbf{Ker}(\Psi_Y) > 6$. Since $\mathbf{Ker}(\Psi_Y) \subset \text{Ad}(k)\mathfrak{m}_2$ and since $\dim \mathfrak{m}_2 = 7$, we have $\mathbf{Ker}(\Psi_Y) = \text{Ad}(k)\mathfrak{m}_2$ and hence $\dim \mathbf{Ker}(\Psi_Y) = 7$. This proves (3). \square

Corollary 13. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0 \neq 0$) and $Y_1 \in \mathfrak{m}_1$ ($Y_1 \neq 0$). Then:*

- (1) $\mathbf{Ker}(\Psi_{Y_0}) \subset \{Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y'_0, Y_0) = 0\}$. In particular, if Y_0 is singular with respect to Ψ , then $\mathbf{Ker}(\Psi_{Y_0}) = \{Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y'_0, Y_0) = 0\}$.
- (2) $\mathbf{Ker}(\Psi_{Y_1}) \subset \{Y'_1 \in \mathfrak{m}_1 \mid (Y'_1, Y_1) = 0\}$. In particular, if Y_1 is singular with respect to Ψ , then $\mathbf{Ker}(\Psi_{Y_1}) = \{Y'_1 \in \mathfrak{m}_1 \mid (Y'_1, Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0 \neq 0$). By Proposition 9 (1), we know that there is an element $k_0 \in K$ satisfying (4.2). Applying Proposition 12 to Y_0 , we easily get $\mathbf{Ker}(\Psi_{Y_0}) \subset \{Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y'_0, Y_0) = 0\}$. Assume that Y_0 is singular with respect to Ψ . Then, by the equality $\mathbf{Ker}(\Psi_{Y_0}) = \text{Ad}(k_0)\mathfrak{m}_2$, we get (1).

The assertion (2) is similarly dealt with. \square

Let $\Psi \in S^2\mathfrak{m}^* \otimes \mathbf{N}$. A subspace U of \mathfrak{m} is called *singular* with respect to Ψ if each element of U is singular with respect to Ψ .

Proposition 14. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Let $Y \in \mathfrak{m}$ ($Y \neq 0$) and let $k \in K$ satisfy $\text{Ad}(k)\mu \in \mathbf{R}Y$. Assume that Y is non-singular with respect to Ψ . Then:*

- (1) $\mathbf{Ker}(\Psi_Y)$ is a singular subspace with respect to Ψ .
- (2) There is an element $Y' \in \text{Ad}(k)\mathfrak{m}_2$ satisfying $\Psi(Y, Y') \neq 0$ and

$$\mathbf{N} = \mathbf{R}\Psi(Y, Y') + \Psi_{Y''}(\mathfrak{m}) \text{ (orthogonal direct sum)}, \quad (5.1)$$

where Y'' is an arbitrary non-zero element of $\mathbf{Ker}(\Psi_Y)$.

Proof. Since Y is non-singular with respect to Ψ , we have $\mathbf{Ker}(\Psi_Y) \subsetneq \text{Ad}(k)\mathfrak{m}_2$ (see Proposition 12). Take a non-zero element $Y' \in \text{Ad}(k)\mathfrak{m}_2$ such that $(Y', \mathbf{Ker}(\Psi_Y)) = 0$. Then, since $Y' \notin \mathbf{Ker}(\Psi_Y)$, we have $\Psi(Y, Y') \neq 0$.

Let $Y'' \in \mathbf{Ker}(\Psi_Y)$ ($Y'' \neq 0$). Then, by the Gauss equation (2.1) we have

$$([\![Y', Y'']\!], Y], W) = \langle \Psi(Y', Y), \Psi(Y'', W) \rangle - \langle \Psi(Y', W), \Psi(Y'', Y) \rangle, \quad (5.2)$$

where W is an arbitrary element of \mathfrak{m} . Note that the left hand side of (5.2) vanishes, because

$$([\![Y', Y'']\!], Y] \in [[\text{Ad}(k)\mathfrak{m}_2, \text{Ad}(k)\mathfrak{m}_2], \text{Ad}(k)\mu] = \text{Ad}(k)[[\mathfrak{m}_2, \mathfrak{m}_2], \mu] = 0.$$

We also note that $\Psi(Y'', Y) = 0$, because $Y'' \in \mathbf{Ker}(\Psi_Y)$. Consequently, we have $\langle \Psi(Y', Y), \Psi(Y'', W) \rangle = 0$. This implies that each element of $\Psi_{Y''}(\mathfrak{m})$ is orthogonal to $\Psi(Y', Y)$. Therefore, $\Psi_{Y''}(\mathfrak{m}) \neq \mathbf{N}$, implying that Y'' is singular with respect to Ψ . Hence, by Proposition 12 (3) we have $\dim \Psi_{Y''}(\mathfrak{m}) = 9$, which proves (5.1). \square

The following lemma assures that for each $\Psi \in \mathcal{G}_o(\mathbf{N})$ there are many high dimensional singular subspaces with respect to Ψ .

Lemma 15. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Then, there are singular subspaces U and V with respect to Ψ satisfying $U \subset \mathfrak{a} + \mathfrak{m}_2$, $V \subset \mathfrak{m}_1$, $\dim U \geq 6$ and $\dim V \geq 6$.*

Proof. If $\mathfrak{a} + \mathfrak{m}_2$ contains no non-singular element with respect to Ψ , then we can take $U = \mathfrak{a} + \mathfrak{m}_2$. (Note that $\dim(\mathfrak{a} + \mathfrak{m}_2) = 8$.) On the contrary, if $\mathfrak{a} + \mathfrak{m}_2$ contains a non-singular element Y_0 , then we set $U = \mathbf{Ker}(\Psi_{Y_0})$. Then, we know that $U \subset \mathfrak{a} + \mathfrak{m}_2$, $\dim U = 6$ (see Proposition 12 (2) and Corollary 13 (1)) and that U is a singular subspace with respect to Ψ (see Proposition 14 (2)). Similarly, we can select a singular subspace $V \subset \mathfrak{m}_1$ with $\dim V \geq 6$. \square

Proposition 16. *Let $\Psi \in \mathcal{G}_o(\mathbf{N})$. Let U and V be arbitrary singular subspaces with respect to Ψ satisfying $U \subset \mathfrak{a} + \mathfrak{m}_2$, $V \subset \mathfrak{m}_1$ and $\dim U \geq 6$, $\dim V \geq 6$. Then there are two vectors \mathbf{A} and $\mathbf{B} \in \mathbf{N}$ satisfying:*

- (1) $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$;
- (2) $\Psi(\xi, Y_0) = (\xi, Y_0)\mathbf{A}$, $\forall \xi \in U, \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$;
- (3) $\Psi(\eta, Y_1) = (\eta, Y_1)\mathbf{B}$, $\forall \eta \in V, \forall Y_1 \in \mathfrak{m}_1$;
- (4) $\langle \mathbf{A}, \Psi_{Y_0}(\mathfrak{m}_1) \rangle = \langle \mathbf{B}, \Psi_{Y_0}(\mathfrak{m}_1) \rangle = 0$, $\forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.

Proof. Let $\xi \in U$ ($\xi \neq 0$). Since ξ is singular with respect to Ψ , $\mathbf{Ker}(\Psi_\xi)$ coincides with the orthogonal complement of $\mathbf{R}\xi$ in $\mathfrak{a} + \mathfrak{m}_2$ (see Corollary 13 (1)). Hence, the equality $\Psi(\xi, Y_0) = 0$ holds for each $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfying $(\xi, Y_0) = 0$. In particular, we have

$$\Psi(\xi, \xi') = 0, \quad \forall \xi, \xi' \in U \text{ with } (\xi, \xi') = 0.$$

Then, applying the same argument as in the proof of Proposition 9 of [7], we can prove that there is a vector $\mathbf{A} \in \mathbf{N}$ satisfying

$$\Psi(\xi, \xi') = (\xi, \xi')\mathbf{A}, \quad \forall \xi, \xi' \in U. \quad (5.3)$$

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $(Y_0, U) = 0$. Then, since $(\xi, Y_0) = 0$, we have $\Psi(\xi, Y_0) = 0$ and $(\xi, Y_0)\mathbf{A} = 0$. This together with (5.3) proves (2). The assertion (3) can be proved in the same way.

We now prove (1). Let $\xi, \xi' \in U$ satisfy $(\xi, \xi') = 0$ and $(\xi, \xi) = (\xi', \xi') = 1$. Put $X = Z = \xi$ and $Y = W = \xi'$ into the Gauss equation (2.1). Then, we have

$$([\xi, \xi'], \xi, \xi') = \langle \Psi(\xi, \xi), \Psi(\xi', \xi') \rangle - \langle \Psi(\xi, \xi'), \Psi(\xi', \xi) \rangle.$$

Since $[[\xi, \xi'], \xi] = 4(\mu, \mu)\xi'$ (see (4.4)), $\Psi(\xi, \xi) = \Psi(\xi', \xi') = \mathbf{A}$ and $\Psi(\xi, \xi') = 0$, we have $\langle \mathbf{A}, \mathbf{A} \rangle = 4(\mu, \mu)$. Similarly, by (4.7) we can prove $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$, proving (1).

Finally, we prove (4). Let $Y_1 \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Take an element $\xi \in U$ satisfying $(\xi, Y_0) = 0$ and $(\xi, \xi) = 1$. Such ξ can exist, because $\dim U \geq 6$. Put $X = Z = \xi$, $Y = Y_0$ and $W = Y_1$ into the Gauss equation (2.1). Then we have

$$([\xi, Y_0], \xi, Y_1) = \langle \Psi(\xi, \xi), \Psi(Y_0, Y_1) \rangle - \langle \Psi(\xi, Y_1), \Psi(Y_0, \xi) \rangle.$$

Since $(\xi, Y_0) = 0$, we have $\Psi(\xi, Y_0) = 0$ and $[[\xi, Y_0], \xi] = 4(\mu, \mu)Y_0$ (see (4.4)). Moreover, since $\Psi(\xi, \xi) = \mathbf{A}$ and $(Y_0, Y_1) = 0$, we have

$$\begin{aligned} \langle \mathbf{A}, \Psi_{Y_0}(Y_1) \rangle &= \langle \Psi(\xi, \xi), \Psi(Y_0, Y_1) \rangle \\ &= \langle \Psi(\xi, Y_1), \Psi(Y_0, \xi) \rangle + 4(\mu, \mu)(Y_0, Y_1) \\ &= 0. \end{aligned}$$

Since Y_1 is an arbitrary element of \mathfrak{m}_1 , we have $\langle \mathbf{A}, \Psi_{Y_0}(\mathfrak{m}_1) \rangle = 0$. In a similar way, the equality $\langle \mathbf{B}, \Psi_{Y_0}(\mathfrak{m}_1) \rangle = 0$ can be proved. \square

Remark 4. As seen in the proof of Lemma 15, singular subspaces U and V may not be uniquely determined. However, it is noted that the vectors \mathbf{A} and \mathbf{B} in Proposition 16 do not depend on the choice of U and V . In fact, let U' and V' be different singular subspaces with respect to Ψ satisfying $U' \subset \mathfrak{a} + \mathfrak{m}_2$ and $V' \subset \mathfrak{m}_1$ with $\dim U' \geq 6$, $\dim V' \geq 6$. Let \mathbf{A}' and \mathbf{B}' be vectors of \mathbf{N} satisfying (1) \sim (4) of Proposition 16. Then, since $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1 = 8$, we have $U \cap U' \neq 0$, $V \cap V' \neq 0$. Take $\xi \in U \cap U'$ and $\eta \in V \cap V'$ such that $(\xi, \xi) = (\eta, \eta) = 1$. Then we have $\mathbf{A} = \Psi(\xi, \xi) = \mathbf{A}'$ and $\mathbf{B} = \Psi(\eta, \eta) = \mathbf{B}'$, showing our assertion.

In the following discussions, we fix an element $\Psi \in \mathcal{G}_o(\mathbf{N})$, singular subspaces U, V and vectors \mathbf{A}, \mathbf{B} stated in Proposition 16 and prove several lemmas which are indispensable to the proof of Theorem 11.

Lemma 17. *Let $\xi \in U, \eta \in V, Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Set $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$. Then $C > 0$ and:*

- (1) $\langle \Psi_{Y_0}(\eta), \Psi_{Y_0}(Y_1) \rangle = \{ \langle \Psi(Y_0, Y_0), \mathbf{B} \rangle - (\mu, \mu)(Y_0, Y_0) \} (\eta, Y_1)$;
- (2) $\langle \Psi_\xi(\eta), \Psi_\xi(Y_1) \rangle = C(\xi, \xi)(\eta, Y_1)$.

Proof. Putting $X = Z = Y_0, Y = Y_1$ and $W = \eta$ into (2.1), we have

$$([[Y_0, Y_1], Y_0], \eta) = \langle \Psi(Y_0, Y_0), \Psi(Y_1, \eta) \rangle - \langle \Psi(Y_0, \eta), \Psi(Y_1, Y_0) \rangle.$$

Since $[[Y_0, Y_1], Y_0] = (\mu, \mu)(Y_0, Y_0)Y_1$ (see (4.5)) and $\Psi(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we easily get (1). Putting $Y_0 = \xi \in U$ into (1), we easily have (2). If we set $Y_1 = \eta \in V$ in (2), we have $\langle \Psi_\xi(\eta), \Psi_\xi(\eta) \rangle = C(\xi, \xi)(\eta, \eta)$. Since $\mathbf{Ker}(\Psi_\xi) \cap \mathfrak{m}_1 = 0$ (see Corollary 13 (1)), we have $\Psi_\xi(\eta) \neq 0$ if $\eta \neq 0$. Consequently, we have $C > 0$. \square

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Let ξ^0 be a non-zero element of U satisfying $(\xi^0, Y_0) = 0$. (Such ξ^0 exists, because $\dim U \geq 6$.) We define a linear mapping $\Theta_{Y_0, \xi^0} : V \rightarrow \mathbf{N}$ by

$$\Theta_{Y_0, \xi^0}(\eta) = \Psi_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \Psi_{\xi^0}([[\xi^0, \eta], Y_0]), \quad \eta \in V.$$

Then we have

Lemma 18. $\langle \mathbf{A}, \Theta_{Y_0, \xi^0}(V) \rangle = \langle \Psi_{\xi^0}(V), \Theta_{Y_0, \xi^0}(V) \rangle = 0$.

Proof. We first note that $[[\xi^0, \eta], Y_0] \in \mathfrak{m}_1$ for $\eta \in V$ and note that $\Theta_{Y_0, \xi^0}(V) \subset \Psi_{Y_0}(\mathfrak{m}_1) + \Psi_{\xi^0}(\mathfrak{m}_1)$. By Proposition 16 (4), we have $\langle \mathbf{A}, \Psi_{Y_0}(\mathfrak{m}_1) \rangle = \langle \mathbf{A}, \Psi_{\xi^0}(\mathfrak{m}_1) \rangle = 0$ and hence $\langle \mathbf{A}, \Theta_{Y_0, \xi^0}(V) \rangle = 0$.

Let $\eta, \eta' \in V$. Then by putting $X = Y_0$, $Y = \eta'$, $Z = \eta$ and $W = \xi^0$ into the Gauss equation (2.1), we have

$$\begin{aligned} ([Y_0, \eta'], \eta), \xi^0) &= \langle \Psi(Y_0, \eta), \Psi(\eta', \xi^0) \rangle - \langle \Psi(Y_0, \xi^0), \Psi(\eta', \eta) \rangle \\ &= \langle \Psi_{Y_0}(\eta), \Psi_{\xi^0}(\eta') \rangle - \langle \mathbf{A}, \mathbf{B} \rangle(Y_0, \xi^0)(\eta', \eta). \end{aligned}$$

Since $(Y_0, \xi^0) = 0$, we have

$$\langle \Psi_{Y_0}(\eta), \Psi_{\xi^0}(\eta') \rangle = ([Y_0, \eta'], \eta), \xi^0). \quad (5.4)$$

On the other hand, we have $\langle \Psi_{\xi^0}([[\xi^0, \eta], Y_0]), \Psi_{\xi^0}(\eta') \rangle = C(\xi^0, \xi^0)([[\xi^0, \eta], Y_0], \eta')$ (see Lemma 17 (2)). Therefore,

$$\begin{aligned} \langle \Theta_{Y_0, \xi^0}(\eta), \Psi_{\xi^0}(\eta') \rangle &= \langle \Psi_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \Psi_{\xi^0}([[\xi^0, \eta], Y_0]), \Psi_{\xi^0}(\eta') \rangle \\ &= ([Y_0, \eta'], \eta), \xi^0) + ([\xi^0, \eta], Y_0), \eta') \\ &= -([Y_0, \eta'], [\xi^0, \eta]) + ([\xi^0, \eta], [Y_0, \eta']) \\ &= 0. \end{aligned}$$

This completes the proof. \square

We can further show

Lemma 19. *Let $\eta \in V$. Assume that $[[\xi^0, \eta], Y_0] \in V$. Then:*

$$|\Theta_{Y_0, \xi^0}(\eta)|^2 = [\langle \Psi(Y_0, Y_0), \mathbf{B} \rangle - (\mu, \mu)(Y_0, Y_0) \{1 + (\mu, \mu)/C\}] (\eta, \eta). \quad (5.5)$$

Proof. Set $\eta' = [[\xi^0, \eta], Y_0]$. By Lemma 18, Lemma 17 and the equality (5.4) we have

$$\begin{aligned} \langle \Theta_{Y_0, \xi^0}(\eta), \Theta_{Y_0, \xi^0}(\eta) \rangle &= \langle \Psi_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \Psi_{\xi^0}(\eta'), \Theta_{Y_0, \xi^0}(\eta) \rangle \\ &= \langle \Psi_{Y_0}(\eta), \Theta_{Y_0, \xi^0}(\eta) \rangle \\ &= \langle \Psi_{Y_0}(\eta), \Psi_{Y_0}(\eta) \rangle + \frac{1}{C(\xi^0, \xi^0)} \langle \Psi_{Y_0}(\eta), \Psi_{\xi^0}(\eta') \rangle \\ &= \{ \langle \Psi(Y_0, Y_0), \mathbf{B} \rangle - (\mu, \mu)(Y_0, Y_0) \} (\eta, \eta) \\ &\quad + \frac{1}{C(\xi^0, \xi^0)} ([Y_0, \eta'], \eta), \xi^0). \end{aligned}$$

Since $[\xi^0, \eta] \in \mathfrak{k}_1$, by (4.8) and (4.5) we have

$$\begin{aligned}
([\![Y_0, \eta']\!, \eta], \xi^0) &= -([Y_0, \eta'], [\xi^0, \eta]) \\
&= (Y_0, [[\xi^0, \eta], \eta']) \\
&= (Y_0, [[\xi^0, \eta], [[\xi^0, \eta], Y_0]]) \\
&= -(\mu, \mu)([\xi^0, \eta], [\xi^0, \eta])(Y_0, Y_0) \\
&= (\mu, \mu)([\xi^0, [\xi^0, \eta]], \eta)(Y_0, Y_0) \\
&= -(\mu, \mu)^2(\xi^0, \xi^0)(\eta, \eta)(Y_0, Y_0).
\end{aligned}$$

Therefore, we obtain (5.5). \square

Lemma 20. *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then:*

(1) $\langle \Psi(Y_0, Y_0), \mathbf{B} \rangle = (\mu, \mu)(Y_0, Y_0) \{1 + (\mu, \mu)/C\}$.

(2) *Let ξ^0 be a non-zero element of U satisfying $(Y_0, \xi^0) = 0$. Then, $\Theta_{Y_0, \xi^0}(\eta) = 0$, i.e., the equality*

$$\Psi(Y_0, \eta) + \frac{1}{C(\xi^0, \xi^0)} \Psi(\xi^0, [[\xi^0, \eta], Y_0]) = 0 \quad (5.6)$$

holds for each $\eta \in V$ satisfying $[[\xi^0, \eta], Y_0] \in V$.

Proof. We first show that there is a non-zero element $\eta^0 \in V$ satisfying $\Theta_{Y_0, \xi^0}(\eta^0) = 0$ and $[[\xi^0, \eta^0], Y_0] \in V$. Let \mathbf{D} be the orthogonal complement of $\mathbf{RA} + \Psi_{\xi^0}(V)$ in \mathbf{N} and let V' be the orthogonal complement of V in \mathfrak{m}_1 . By Lemma 18, we easily have $\Theta_{Y_0, \xi^0}(V) \subset \mathbf{D}$. Therefore, to obtain η^0 satisfying the above condition, it suffices to find a non-zero solution $\eta = \eta^0 \in V$ of the system of linear equations

$$\langle \Theta_{Y_0, \xi^0}(\eta), \mathbf{D} \rangle = ([[\xi^0, \eta], Y_0], V') = 0. \quad (5.7)$$

Since $\mathbf{Ker}(\Psi_{\xi^0}) \cap \mathfrak{m}_1 = 0$ (see Corollary13 (1)) and $\langle \mathbf{A}, \Psi_{\xi^0}(\mathfrak{m}_1) \rangle = 0$ (see Proposition 16 (4)), we have $\dim(\mathbf{RA} + \Psi_{\xi^0}(V)) = 1 + \dim V \geq 7$. (Recall that we are assuming $V \subset \mathfrak{m}_1$ and $\dim V \geq 6$.) Hence, we have $\dim \mathbf{D} \leq \dim \mathbf{N} - 7 = 3$. Moreover, we have $\dim V' = 8 - \dim V \leq 2$. Consequently, (5.7) is the system of linear equations on V ($\dim V \geq 6$) composed of at most 5 linearly independent equations. Therefore, we can find a non-zero solution $\eta^0 \in V$ of (5.7). Putting $\eta = \eta^0$ into (5.5), we obtain the equality (1). Further, putting (1) into (5.5), we have $\Theta_{Y_0, \xi^0}(\eta) = 0$ for any $\eta \in V$ satisfying $[[\xi^0, \eta], Y_0] \in V$. \square

Lemma 21. *The vectors \mathbf{A} and \mathbf{B} are linearly independent and $\langle \mathbf{A}, \mathbf{B} \rangle = 2(\mu, \mu)$, $C = (\mu, \mu)$.*

Proof. Let $\xi \in U$ with $(\xi, \xi) = 1$. Since $\Psi(\xi, \xi) = \mathbf{A}$ (see (5.3)), by putting $Y_0 = \xi$ into the equality in Lemma 20 (1), we easily have $\langle \mathbf{A}, \mathbf{B} \rangle = (\mu, \mu) \{1 + (\mu, \mu)/C\}$. Since $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$, it immediately follows that $C^2 = (\mu, \mu)^2$. Since $C > 0$, we get $C = (\mu, \mu)$ and hence $\langle \mathbf{A}, \mathbf{B} \rangle = 2(\mu, \mu)$. This together with Proposition 16 (1) proves that \mathbf{A} and \mathbf{B} are linearly independent. \square

These being prepared, we show Theorem 11.

Proof of Theorem 11. First we show that μ is singular with respect to any element $\Psi \in \mathcal{G}_o(\mathbf{N})$. Suppose that there is an element $\Psi_0 \in \mathcal{G}_o(\mathbf{N})$ such that μ is non-singular with respect to Ψ_0 . Then, $\mathbf{Ker}((\Psi_0)_\mu)$ is a singular subspace with respect to Ψ_0 and it satisfies $\dim \mathbf{Ker}((\Psi_0)_\mu) = 6$ and $\mathbf{Ker}((\Psi_0)_\mu) \subset \mathfrak{m}_2$ (see Proposition 12 and Proposition 14).

Now, set $\Psi = \Psi_0$ and $U = \mathbf{Ker}((\Psi_0)_\mu)$ in Proposition 16. Let \mathbf{A}, \mathbf{B} be the vectors of \mathbf{N} satisfying (1)–(4) of Proposition 16. Let $\xi \in U = \mathbf{Ker}((\Psi_0)_\mu)$ with $\xi \neq 0$. First, we show $\mathbf{B} \in (\Psi_0)_\xi(\mathfrak{m})$. In fact, there is a non-zero element $Y_2^0 \in \mathfrak{m}_2$ satisfying $\Psi_0(\mu, Y_2^0) \neq 0$ and $\mathbf{N} = \mathbf{R}\Psi_0(\mu, Y_2^0) + (\Psi_0)_\xi(\mathfrak{m})$ (orthogonal direct sum) (see Proposition 14). By Lemma 20 (1) and by the relation

$$\Psi_0(\mu, Y_2^0) = \frac{1}{2} (\Psi_0(\mu + Y_2^0, \mu + Y_2^0) - \Psi_0(\mu, \mu) - \Psi_0(Y_2^0, Y_2^0)),$$

we easily have $\langle \Psi_0(\mu, Y_2^0), \mathbf{B} \rangle = 0$, which proves $\mathbf{B} \in (\Psi_0)_\xi(\mathfrak{m})$. Since $(\Psi_0)_\xi(\mathfrak{m}) = \mathbf{R}\mathbf{A} + (\Psi_0)_\xi(\mathfrak{m}_1)$ (orthogonal direct sum) and $\langle \mathbf{B}, (\Psi_0)_\xi(\mathfrak{m}_1) \rangle = 0$ (see Proposition 16 (2), (4)), we have $\mathbf{B} \in \mathbf{R}\mathbf{A}$. This contradicts Lemma 21. Accordingly, we can conclude that μ is singular with respect to any element $\Psi \in \mathcal{G}_o(\mathbf{N})$.

Now we show that any element of \mathfrak{m} is singular with respect to any $\Psi \in \mathcal{G}_o(\mathbf{N})$. Let Y be a non-zero element of \mathfrak{m} . Take an element $k \in K$ such that $\text{Ad}(k)\mu \in \mathbf{R}Y$ and define $\Psi' \in S^2\mathfrak{m}^* \otimes \mathbf{N}$ by

$$\Psi'(Y', Y'') = \Psi(\text{Ad}(k)Y', \text{Ad}(k)Y''), \quad Y', Y'' \in \mathfrak{m}.$$

Then, it is easily seen that $\Psi' \in \mathcal{G}_o(\mathbf{N})$. Applying the arguments developed above, we know that μ is also singular with respect to Ψ' . Note that $\Psi'_\mu(\mathfrak{m}) = \Psi_{\text{Ad}(k)\mu}(\text{Ad}(k)\mathfrak{m}) = \Psi_Y(\mathfrak{m})$. Then, since $\Psi'_\mu(\mathfrak{m}) \neq \mathbf{N}$, we have $\Psi_Y(\mathfrak{m}) \neq \mathbf{N}$, implying that Y is singular with respect to Ψ .

Accordingly, in Proposition 16 and in the discussion after it, we may allow to put $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Therefore, by Proposition 16 and Lemma 21, we get (1) of Theorem 11. Further, putting $Y_0 = Y_2 \in \mathfrak{m}_2$, $\xi^0 = \mu$ and $\eta = Y_1$ into (5.6), we get (2) of Theorem 11. The assertion (3) of Theorem 11 follows from Lemma 17 (2) and Lemma 21. This completes the proof of the theorem. \square

6. PROOF OF THEOREM 10

Let $\{E_i (1 \leq i \leq 8)\}$ be an orthonormal basis of \mathfrak{m}_1 . (Note that $\dim \mathfrak{m}_1 = 8$.) Let $\Psi \in \mathcal{G}_o(\mathbf{N})$ and let \mathbf{A}, \mathbf{B} be the vectors of \mathbf{N} stated in Theorem 11. We define vectors $\{\mathbf{F}_i (1 \leq i \leq 10)\}$ of \mathbf{N} by setting $\mathbf{F}_i = \Psi(\mu, E_i)/(\mu, \mu)$ ($1 \leq i \leq 8$), $\mathbf{F}_9 = (\mathbf{A} + \mathbf{B})/2\sqrt{3}|\mu|$ and $\mathbf{F}_{10} = (\mathbf{A} - \mathbf{B})/2|\mu|$. We now show that $\{\mathbf{F}_i (1 \leq i \leq 10)\}$ forms an orthonormal basis of \mathbf{N} . By Theorem 11 (3) we have $\langle \mathbf{F}_i, \mathbf{F}_j \rangle = \delta_{ij}$ ($1 \leq i, j \leq 8$), where δ_{ij} denotes Kronecker's delta. Moreover, since $\langle \mathbf{A}, \mathbf{F}_i \rangle = \langle \mathbf{B}, \mathbf{F}_i \rangle = 0$ ($1 \leq i \leq 8$) (see Theorem 11 (1d)), we have $\langle \mathbf{F}_9, \mathbf{F}_i \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_i \rangle = 0$ ($1 \leq i \leq 8$). The equalities $\langle \mathbf{F}_9, \mathbf{F}_9 \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_{10} \rangle = 1$ and $\langle \mathbf{F}_9, \mathbf{F}_{10} \rangle = 0$ immediately follow from Theorem 11 (1a).

Now let Ψ' be another element of $\mathcal{G}_o(\mathbf{N})$. Let \mathbf{A}' and \mathbf{B}' be the vectors stated in Theorem 11 for Ψ' . As in the case of Ψ we can also define an orthonormal basis $\{\mathbf{F}'_i (1 \leq i \leq 10)\}$ of \mathbf{N} . Then, there is an element $h \in O(10)$ satisfying $\mathbf{F}'_i = h\mathbf{F}_i$ ($1 \leq i \leq 10$). Here we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\Psi'(\mu, E_i) = h\Psi(\mu, E_i)$ ($1 \leq i \leq 8$). Set $\Phi = \Psi' - h\Psi \in S^2\mathfrak{m}^* \otimes \mathbf{N}$. Then, by Theorem 11 (1) we have

$$\Phi(\mathfrak{a} + \mathfrak{m}_2, \mathfrak{a} + \mathfrak{m}_2) = \Phi(\mathfrak{m}_1, \mathfrak{m}_1) = \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0.$$

By the fact $[[\mu, \mathfrak{m}_1], \mathfrak{m}_2] \subset \mathfrak{m}_1$ and Theorem 11 (2), we have

$$\Phi(\mathfrak{m}_2, \mathfrak{m}_1) \subset \Phi(\mu, [[\mu, \mathfrak{m}_1], \mathfrak{m}_2]) \subset \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0,$$

which proves that $\Phi(\mathfrak{m}_2, \mathfrak{m}_1) = 0$. Therefore, we have $\Phi = 0$, i.e., $\Psi' = h\Psi$. This implies that the Gaussain variety $\mathcal{G}_o(\mathbf{N})$ is EOS. This completes the proof of Theorem 10. \square

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