# Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent

Toshihide Futamura and Yoshihiro Mizuta

**Abstract**

Our aim in this paper is to deal with 0-Hölder continuity for Riesz potentials of functions belonging to Lebesgue's $L^p$ space of variable exponent, in the borderline case of Sobolev's theorem. We are also concerned with exponential integrability for Riesz potentials.

## 1 Introduction

Let $\mathbf{R}^n$ denote the $n$-dimensional Euclidean space. We consider the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $\mathbf{R}^n$, which is defined by

$$U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy.$$

Here $0 < \alpha < n$. Following Kováčik and Rákosník [9], we consider a positive continuous function $p(\cdot) : \mathbf{R}^n \to [1, \lambda)$, $1 < \lambda < \infty$, and a measurable function $f$ satisfying

$$\int |f(y)|^{p(y)} dy < \infty.$$

Recently Diening [3] has established embedding results for Riesz potentials of such functions. For related results, see also Edmunds-Rákosník [4], Futamura-Mizuta-Shimomura [6] and Růžička [13]. In these discussions, the continuity of Hardy-Littlewood maximal functions is a crucial tool (see Diening [2]).

In case $p(\cdot)$ is a constant $p_0$ and $p_0 > n/\alpha$, well known Sobolev's theorem says that $U_\alpha f$ is continuous in $\mathbf{R}^n$ (see e.g. [1], [10], [12]). Our first aim in this paper is to discuss the continuity for $\alpha$-potentials of functions in $L^{p(\cdot)}$ spaces when $p(x) \geq n/\alpha$ for $x \in \mathbf{R}^n$ and $p(\cdot)$ satisfies a so called 0-Hölder condition, as an extension of Harjulehto-Hästö [7].

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We also study exponential integrabilities of $\alpha$-potentials when they are not continuous, as an extension of Trudinger's exponential integrability (see Hedberg [8] and Adams-Hedberg [1]).

## 2 Continuity of potentials

Throughout this paper, let $C$ denote various constants independent of the variables in question.

Let $G$ be a bounded open set in $\mathbf{R}^n$ and $B(x_0, r_0) \subset G$, where $B(x_0, r_0)$ denotes the open ball centered at $x_0$ of radius $r_0 > 0$. Consider a positive continuous function $p(\cdot)$ on $G$. In this and the next sections let us assume that :

(p1) $\inf_{G \setminus B(x_0, r_0)} p(x) > p_- = n/\alpha$ and $p_+ = \sup_G p(x) < \infty$;

(p2) $$p(y) \geq p_- + \frac{a \log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)} + \frac{\tilde{a}}{\log(1/|x_0 - y|)} \qquad \text{for } y \in B(x_0, r_0)\ ,$$

where $a \geq 0$ and $\tilde{a}$ is a real number. In our discussions we may assume that

(p3) $$p(y) \leq p_- + \frac{a \log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)} + \frac{C}{\log(1/|x_0 - y|)}$$

for $y \in B(x_0, r_0)$ .

Set

$$\omega_{a',a''}(r) = \frac{a' \log(\log(1/r))}{\log(1/r)} + \frac{a''}{\log(1/r)}$$

and $\omega_{a',a''}(0) = 0$. If $a' > 0$ or $a' = 0$ and $a'' > 0$, then we can find $r^* > 0$ so small that $\omega_{a',a''}$ is nondecreasing on the interval $[0, 2r^*]$ and

$$\omega_{a',a''}(s + t) \leq \omega_{a',a''}(s) + \omega_{a',a''}(t) \tag{1}$$

for $0 \leq s \leq t \leq r^*$.

Let $1/p'(x) = 1 - 1/p(x)$ and $1/p'_- = 1 - 1/p_-$.

We begin with the following result.

Lemma 2.1. *If $a > (n - \alpha)/\alpha^2$, then*

$$\int_{G \cap B(x,\delta)} |x - y|^{p'(y)(\alpha - n)} dy \leq C(\log(1/\delta))^{1 - a\alpha^2/(n-\alpha)}$$

*for all $x \in G$ and $\delta \in (0, 2^{-1})$.*

Proof. First note that

$$p'(y) - p'_- = -\frac{p(y) - p_-}{(p(y) - 1)(p_- - 1)} = -\frac{p(y) - p_-}{(p_- - 1)^2} + \frac{(p(y) - p_-)^2}{(p(y) - 1)(p_- - 1)^2}.$$

Since $p_- - 1 = (n-\alpha)/\alpha$, by conditions on $p$, we can find $C > 0$ so that

$$p'(y) \le p'_- - \omega_{a',-C}(|x_0 - y|) \qquad (a' = a\alpha^2/(n-\alpha)^2) \tag{2}$$

for all $y \in B(x_0, r_0)$. For simplicity, set

$$\omega(r) = \omega_{a',-C}(r) = \frac{a\alpha^2}{(n-\alpha)^2}\frac{\log(\log(1/r))}{\log(1/r)} - \frac{C}{\log(1/r)}.$$

Noting that $\omega$ is nondecreasing and doubling on $[0, r_0]$ by (1), we have for $0 < \delta \le |x_0 - x|/2$ and $x \in B(x_0, r_0/2)$

$$\begin{aligned}
\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}dy &\le \sum_j \int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} |x-y|^{p'(y)(\alpha-n)}dy \\
&\le \sum_j (2^{-j}\delta)^{(\alpha-n)(p'_- - \omega(2^{-j}\delta))}\sigma_n(2^{-j+1}\delta)^n \\
&\le C\sum_j (2^{-j}\delta)^{-(\alpha-n)\omega(2^{-j}\delta)} \\
&\le C\sum_j (\log 1/(2^{-j}\delta))^{-a\alpha^2/(n-\alpha)} \\
&\le C\int_0^\delta (\log(1/t))^{-a\alpha^2/(n-\alpha)}t^{-1}dt \\
&= C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)},
\end{aligned}$$

since $a > (n-\alpha)/\alpha^2$, where $\sigma_n$ denotes the volume of the unit ball. If $y \in G \setminus B(x, |x_0 - x|/2)$, then $|x_0 - y| \le 3|x - y|$, so that

$$\begin{aligned}
\int_{B(x,\delta)\setminus B(x,|x_0-x|/2)} |x-y|^{p'(y)(\alpha-n)}dy &\le C\int_{G\cap B(x_0,3\delta)} |x_0-y|^{p'(y)(\alpha-n)}dy \\
&\le C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)}
\end{aligned}$$

when $|x_0 - x|/2 \le \delta \le r_0/4$. Therefore it follows that

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}dy \le C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)}$$

for $0 < \delta < 1/2$ and $x \in B(x_0, r_0/2)$.

Noting from condition (p1) that $p_0 = \inf_{y\in G\setminus B(x_0,r_0/4)} p(y) > n/\alpha$, we see that

$$\int_{G\cap B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}dy \le C\delta^{(\alpha p_0 - n)/(p_0-1)}$$

for $x \in G \setminus B(x_0, r_0/2)$ and $\delta > 0$.

Now the proof is completed. □

Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} < \infty$.

THEOREM 2.2. *Let $f$ be a nonnegative measurable function on a bounded open set $G$ with $\|f\|_{p(\cdot)} \le 1$. If $a > (n-\alpha)/\alpha^2$, then $U_\alpha f$ is continuous in $G$. Further,*

$$|U_\alpha f(x) - U_\alpha f(z)| \le C(\log(1/|x-z|))^{-A}$$

*whenever $x, z \in G$ and $|x-z| < 1/2$, where $A = (a\alpha^2/(n-\alpha) - 1)/p'_-$.*

REMARK 2.3. In view of Sobolev's theorem, we see that $U_\alpha f$ is continuous in $G \setminus \{x_0\}$. Harjulehto-Hästö [7] have also discussed the continuity of Sobolev functions.

PROOF OF THEOREM 2.2. First note that

$$\int_G f(y)^{p(y)} dy \le 1 \tag{3}$$

since $\|f\|_{p(\cdot)} \le 1$ by the assumption. Then, for $0 < \mu < 1$, we have by Young's inequality and Lemma 2.1

$$\begin{aligned} \int_{G\cap B(x,\delta)} |x-y|^{\alpha-n} f(y) dy &\le \mu \int_{G\cap B(x,\delta)} \left\{(|x-y|^{\alpha-n}/\mu)^{p'(y)} + f(y)^{p(y)}\right\} dy \\ &\le \mu \left(\mu^{-p'_-} \int_{G\cap B(x,\delta)} |x-y|^{(\alpha-n)p'(y)} dy + 1\right) \\ &\le \mu \left(C\mu^{-p'_-} (\log(1/\delta))^{1-a\alpha^2/(n-\alpha)} + 1\right) \end{aligned}$$

whenever $x \in G$ and $0 < \delta < 1/2$. Now, considering $\mu$ such that $\mu^{p'_-} = (\log(1/\delta))^{1-a\alpha^2/(n-\alpha)}$, we find

$$\int_{G\cap B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C(\log(1/\delta))^{-A}. \tag{4}$$

Hence, if $x, z \in G$ and $|x-z| < 1/4$, then we have

$$\int_{G\cap B(x,2|x-z|)} |x-y|^{\alpha-n} f(y) dy \le C(\log(1/|x-z|))^{-A}.$$

On the other hand we find

$$\begin{aligned} &\int_{G\setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y) dy \\ \le\ & C|x-z| \int_{G\setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y) dy. \end{aligned}$$

This can be estimated along the same lines as above. For simplicity set $\delta = 2|x-z| < 1/2$. Then, for $\mu \ge 1$, letting

$$E = \{y \in G \setminus B(x, 2|x-z|) : |x-y|^{\alpha-n-1} \ge \mu\},$$

we have by Young's inequality and (2)

$$\begin{aligned}
&\int_{G\setminus\{B(x_0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1} f(y)dy \\
\le\ & \mu \int_{G\setminus\{B(x_0,\delta)\cup B(x,\delta)\}} \left\{(|x-y|^{\alpha-n-1}/\mu)^{p'(y)} + f(y)^{p(y)}\right\} dy \\
\le\ & C\mu\left(\int_{E\setminus B(x_0,\delta)} (|x-y|^{\alpha-n-1}/\mu)^{p'(y)} dy + 1\right) \\
\le\ & C\mu\left(\int_{E\setminus B(x_0,\delta)} (|x-y|^{\alpha-n-1}/\mu)^{p'_- - \omega(\delta)} dy + 1\right) \\
\le\ & C\mu\left(\mu^{-p'_- + \omega(\delta)} \int_{G\setminus B(x,\delta)} |x-y|^{(\alpha-n-1)(p'_- - \omega(\delta))} dy + 1\right) \\
\le\ & C\mu\left(\mu^{-p'_- + \omega(\delta)} \delta^{(\alpha-n-1)(p'_- - \omega(\delta))+n} + 1\right) \\
\le\ & C\mu\left(\mu^{-p'_- + \omega(\delta)} \delta^{-p'_-} (\log(1/\delta))^{(\alpha-n-1)a\alpha^2/(n-\alpha)^2} + 1\right).
\end{aligned}$$

Now, considering $\mu$ such that $\mu = \delta^{-1}(\log(1/\delta))^{-a\alpha^2/\{p'_-(n-\alpha)\}}$, we find

$$\int_{G\setminus\{B(x_0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1} f(y)dy \le C\delta^{-1}(\log(1/\delta))^{-a\alpha^2/\{p'_-(n-\alpha)\}}.$$

Further, we obtain by (4)

$$\begin{aligned}
\int_{G\cap B(x_0,\delta)\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y)dy &\le \delta^{-1} \int_{G\cap B(x_0,\delta)} |x_0-y|^{\alpha-n} f(y)dy \\
&\le C\delta^{-1}(\log(1/\delta))^{-A}.
\end{aligned}$$

Therefore it follows that

$$\int_{G\setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \le C(\log(1/|x-z|))^{-A}.$$

Now we establish

$$\begin{aligned}
&|U_\alpha f(x) - U_\alpha f(z)| \\
\le\ & \int_{G\cap B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy + \int_{G\cap B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy \\
& + \int_{G\setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \\
\le\ & C(\log(1/|x-z|))^{-A},
\end{aligned}$$

as required. $\square$

Corollary 2.4. *Suppose*

$$p(x) = p(x_1, ..., x_n) \geq n/\alpha + \frac{a \log(e + \log(1/|x_n|))}{\log(e/|x_n|)}$$

*for* $a > (n-\alpha)/\alpha^2$. *Let* $f$ *be a nonnegative measurable function on* $B = B(0,1)$ *with* $\|f\|_{p(\cdot),B} \leq 1$. *Then* $U_\alpha f$ *is continuous in* $B$ *and it satisfies*

$$|U_\alpha f(x) - U_\alpha f(z)| \leq C(\log(1/|x-z|))^{-A}$$

*whenever* $x, z \in B(0, 1/2)$ *and* $|x - z| < 1/2$, *where* $A = (a\alpha^2/(n-\alpha) - 1)/p'_-$.

Proof. According to the proof of Theorem 2.2, it suffices to show that

$$\int_{B(x,r)} |x-y|^{(\alpha-n)p'(y)} dy \leq C(\log(1/r))^{1-a\alpha^2/(n-\alpha)} \tag{5}$$

for $0 < r < 1/2$ and $|x| < 1/2$. To show this, we may assume that

$$p'(y) \leq p'_- - \omega(|y_n|) \qquad \text{for } y \in B,$$

where $\omega(r) = (a\alpha^2/(n-\alpha)^2) \log(\log(1/r))/\log(1/r) - C/\log(1/r)$ for $0 < r \leq r_0$ and $\omega(r) = \omega(r_0)$ for $r > r_0$. Then, by use of Lemma 2.1, we have

$$\begin{aligned} \int_{B(x,r)} |x-y|^{(\alpha-n)p'(y)} dy &\leq C \int_{\{y_n : |x_n - y_n| < r\}} |x_n - y_n|^{-1+(n-\alpha)\omega(|y_n|)} dy_n \\ &\leq C(\log(1/r))^{1-a\alpha^2/(n-\alpha)}. \end{aligned}$$

Thus (5) holds, and the proof is completed. $\square$

Remark 2.5. Let $b > (a+1)/n > 1$, $0 < r_0 < 1/e$ and

$$p(y) = n + \frac{a \log(\log(1/|y|))}{\log(1/|y|)}$$

for $y \in B(0, r_0)$. Consider the function

$$f(y) = |y|^{-1}(\log(1/|y|))^{-b}$$

for $y \in B(0, r_0)$ and $f = 0$ on $\mathbf{R}^n \setminus B(0, r_0)$. Then we easily see that

$$\int_{B(0,r_0)} |x-y|^{1-n} f(y) dy \geq C(\log(1/|x|))^{1-b} \qquad \text{for } x \in B(0, r_0)$$

and

$$\int_{B(0,r_0)} f(y)^{p(y)} dy \;\leq\; \int_{B(0,r_0)} \{|y|^{-1}(\log(1/|y|))^{-b}\}^n (\log(1/|y|))^a dy < \infty$$

since $-bn + a + 1 < 0$ by our assumption.

This means that the exponent $A$ in Theorem 2.2 is best possible.

REMARK 2.6. Let $a = n - 1$ and

$$p(y) = n + \frac{(n-1)\log(\log(1/|y|))}{\log(1/|y|)}$$

for $y \in B(0, r_0)$. Then there exists a measurable function $f$ on $\mathbf{R}^n$ such that $U_1 f(0) = \infty$ and $\|f\|_{p(\cdot)} < \infty$.

In fact, for $1/n < b \le 1$, consider the function

$$f(y) = |y|^{-1}(\log(1/|y|))^{-1}(\log(\log(1/|y|)))^{-b}$$

for $y \in B(0, r_0)$ and $f = 0$ on $\mathbf{R}^n \setminus B(0, r_0)$. Then we have

$$\int_{B(0,r_0)} |y|^{1-n} f(y) dy = \infty.$$

Since $bn > 1$ by our assumption, we obtain

$$\begin{aligned} \int_{B(0,r_0)} f(y)^{p(y)} dy &\le \int_{B(0,r_0)} \{|y|^{-1}(\log(1/|y|))^{-1}(\log(\log(1/|y|)))^{-b}\}^n (\log(1/|y|))^{n-1} dy \\ &= \int_{B(0,r_0)} |y|^{-n}(\log(1/|y|))^{-1}(\log(\log(1/|y|)))^{-bn} < \infty \end{aligned}$$

In this case we can show exponential integrability (see e.g. [1]), as will be discussed soon.

# 3 Exponential integrability

This section concerns with $p(\cdot)$ such that

$$p(y) \le p_- + \frac{n-\alpha}{\alpha^2} \frac{\log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)}$$

for $y \in B(x_0, r_0)$. In this case, since $\alpha$-potentials of $f \in L^{p(\cdot)}(G)$ may not be continuous, we discuss the exponential integrability of Trudinger type. Our discussions here can be carried out along the same lines as in Hedberg [8].

Before doing so we prepare the following lemma under conditions (p1) and (p2).

LEMMA 3.1. *If $0 < b < a \le (n-\alpha)/\alpha^2$, then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{(\alpha - n)p'(y)} dy \le C(\log(1/\delta))^{1 - b\alpha^2/(n-\alpha)}$$

*for $x \in G$ and $0 < \delta < 1/2$.*

PROOF. For $0 < b < a \le (n-\alpha)/\alpha^2$, set

$$\omega(r) = \frac{b\alpha^2}{(n-\alpha)^2} \frac{\log(\log(1/r))}{\log(1/r)}.$$

As in (2), we can find $r_1 > 0$ such that

$$p'(y) \le p'_- - \omega(|x_0 - y|)$$

for all $y \in B(x_0, r_1)$; in this proof we assume that $r_1 = 4r_0$.

If $x \in B(x_0, 2r_0)$, then we have

$$\begin{aligned}
\int_{B(x,|x_0-x|/2)\setminus B(x,\delta)} |x-y|^{p'(y)(\alpha-n)} dy &\le \sum_j \int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} |x-y|^{p'(y)(\alpha-n)} dy \\
&\le \sum_j (2^{j-1}\delta)^{(\alpha-n)(p'_- - \omega(2^{j-1}\delta))} \sigma_n (2^j\delta)^n \\
&\le C \sum_j (2^{j-1}\delta)^{-(\alpha-n)\omega(2^{j-1}\delta)} \\
&\le C \sum_j (\log 1/(2^{j-1}\delta))^{-b\alpha^2/(n-\alpha)} \\
&\le C \int_\delta^{r_0} (\log(1/t))^{-b\alpha^2/(n-\alpha)} t^{-1} dt \\
&= C(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}.
\end{aligned}$$

If $\delta \ge |x_0 - x|/2$ and $x \in B(x_0, 2r_0)$, then

$$\begin{aligned}
\int_{B(x_0,4r_0)\setminus B(x,\delta)} |x-y|^{p'(y)(\alpha-n)} dy &\le C \int_{B(x_0,4r_0)\setminus B(x,\delta)} |x_0-y|^{p'(y)(\alpha-n)} dy \\
&\le C(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}.
\end{aligned}$$

Finally, since $\inf_{G\setminus B(x_0,r_0)} p(x) > p_- = n/\alpha$, we note that

$$\int_G |x-y|^{p'(y)(\alpha-n)} dy \le C < \infty$$

for $x \in G \setminus B(x_0, 2r_0)$.

Thus the required conclusion follows from these facts. □

If $f$ is a locally integrable function on $G$, then we consider Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\sigma_n r^n} \int_{G\cap B(x,r)} |f(y)| dy.$$

We next prove the estimate of Riesz potentials by use of maximal functions, as in Hedberg [8].

Lemma 3.2. *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\cdot)} \le 1$. If $0 < a \le (n-\alpha)/\alpha^2$ and $A > (1 - a\alpha^2/(n-\alpha))/p'_-$, then*

$$U_\alpha f(x) \le C(\log(Mf(x)+2))^A.$$

Proof. If $(1-a\alpha^2/(n-\alpha))/p'_- < A$, then there exist $0 < b < a$ and $0 < p_0 < p'_-$ such that

$$(1 - b\alpha^2/(n-\alpha))/p_0 < A.$$

We can find $r_1 > 0$ such that $p'(y) > p_0$ for $y \in B(x_0, r_1)$ and

$$\int_{B(x_0,r_1)\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \le C\mu^{-p_0}(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}$$

for $\mu > 1$ and $x \in G$; in this proof, we may assume that $r_1 = 4r_0$. Since (3) holds by the assumption $\|f\|_{p(\cdot)} \le 1$, we have for $\mu > 1$

$$\begin{aligned}
&\int_{B(x_0,4r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \mu\left(\int_{B(x_0,4r_0)\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy + \int_{G\setminus B(x,\delta)} f(y)^{p(y)}dy\right) \\
\le\ & \mu\left(C\mu^{-p_0}(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)} + 1\right).
\end{aligned}$$

Now, considering $\mu$ such that $\mu^{-p_0}(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)} = 1$, we have

$$\int_{B(x_0,4r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C(\log(1/\delta))^\beta$$

with $\beta = \{1 - b\alpha^2/(n-\alpha)\}/p_0$. Since $\inf_{G\setminus B(x_0,r_0)} p(x) > p_- = n/\alpha$, we note that

$$\int_G |x-y|^{\alpha-n} f(y)dy \le C$$

for $x \in G \setminus B(x_0, 2r_0)$. Consequently it follows from [1, (3.1.1)] that

$$\begin{aligned}
U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
&\le C\delta^\alpha Mf(x) + C(\log(1/\delta))^\beta.
\end{aligned}$$

Here, as in the proof of Proposition 3.1.2 in [1], let

$$\delta = (Mf(x))^{-1/\alpha}(\log(Mf(x)+2))^{\beta/\alpha}$$

when $Mf(x)$ is large enough. Then we have

$$U_\alpha f(x) \le C(\log(Mf(x)+2))^\beta \le C(\log(Mf(x)+2))^A,$$

as required. □

By Lemma 3.2 and the fact that $Mf \in L^{p_-}(G)$, we establish the following exponential inequality for $f \in L^{p(\cdot)}(G)$ .

THEOREM 3.3. *For $A > (1-a\alpha^2/(n-\alpha))/p'_- \ge 0$, there exist positive constants $c_1$ and $c_2$ such that*

$$\int_G \exp(c_1(U_\alpha f(x))^{1/A})dx \le c_2$$

*for all nonnegative measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} \le 1$.*

THEOREM 3.4. *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\cdot)} < \infty$. If $A > (1-a\alpha^2/(n-\alpha))/p'_- \ge 0$, then*

$$\int_G \exp(c(U_\alpha f(x))^{1/A})dx < \infty \qquad \text{for all } c > 0.$$

REMARK 3.5. When $a = 0$, Theorems 3.3 and 3.4 hold for $A = 1/p'_- = (n-\alpha)/n$.

# 4 Sobolev's inequality

In this section we are concerned with $p(\cdot)$ satisfying :

(p4) $1 < p_- = \inf_G p(x) \le p(x) < p_+ = \sup_G p(x) = n/\alpha$ ;

(p5) $|p(x)-p(y)| \le \dfrac{\tilde{a}}{\log(1/|x-y|)}$ whenever $|x-y| < 1/2$ ,

for some $\tilde{a} > 0$.

As an example, we may consider the function of the form

$$p(y) = p_0 - \omega(|x_0-y|), \qquad \omega(r) = \frac{\tilde{a}}{\log(1/r)},$$

for $y \in B(x_0, r_0)$ with $r_0$ chosen sufficiently small; set $p(y) = p_+ - \omega(r_0)$ outside $B(x_0, r_0)$. Note here that

$$\omega(s+t) \le \omega(s) + \omega(t)$$

for $0 < s < r_0$ and $0 < t < r_0$ .

Let $1/p^\sharp(x) = 1/p(x) - \alpha/n$.

LEMMA 4.1. *If $\mu > 1$ and $0 < \delta < 1/2$, then*

$$\int_{G\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \le C\left(\mu^{-p'(x)}\frac{\delta^{-q(x)/(p(x)-1)}}{q(x)} + 1\right)$$

*for $x \in G$, where $q(x) = n - \alpha p(x) > 0$.*

PROOF. First find $C > 0$ such that

$$|p'(y) - p'(x)| \le \frac{C}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/2$. Then we have for $\mu > 1$

$$\begin{aligned} & \int_{B(x,\mu^{1/(\alpha-n)})\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \\ \le\ & C\mu^{-p'(x)} \int_{B(x,\mu^{1/(\alpha-n)})\setminus B(x,\delta)} |x-y|^{(\alpha-n)p'(x)} dy \\ \le\ & C\mu^{-p'(x)} \frac{\delta^{p'(x)(\alpha-n/p(x))}}{-p'(x)(\alpha-n/p(x))}, \end{aligned}$$

which yields the required inequality. □

LEMMA 4.2. *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_G |x-y|^{\alpha-n} f(y) dy \le C\tilde{q}(x)^{-\alpha(p(x)-1)/n} Mf(x)^{p(x)/p^\sharp(x)}.$$

*for $x \in G$, where $q(x) = n - \alpha p(x) > 0$ and $\tilde{q}(x) = \min\{q(x), 1\}$.*

PROOF. First consider the case

$$Mf(x)\tilde{q}^{1/p'(x)} > 2^{\alpha+q(x)/p(x)}. \tag{6}$$

Since $\|f\|_{p(\cdot)} \le 1$, we have for $\mu > 1$

$$\begin{aligned} & \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\ \le\ & \mu\left(\int_{G\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + \int_{G\setminus B(x,\delta)} f(y)^{p(y)} dy\right) \\ \le\ & C\mu\left(\mu^{-p'(x)}\frac{\delta^{-q(x)/(p(x)-1)}}{q(x)} + 1\right) \end{aligned}$$

because of Lemma 4.1. Now if we set

$$\mu^{-p'(x)}\frac{\delta^{-q(x)/(p(x)-1)}}{\tilde{q}(x)} = 1,$$

then

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\frac{\delta^{-q(x)/p(x)}}{\tilde{q}(x)^{1/p'(x)}}.$$

It follows from [1, (3.1.1)] that

$$\begin{aligned}\int_G |x-y|^{\alpha-n} f(y)dy &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le C\delta^\alpha Mf(x) + C\frac{\delta^{-q(x)/p(x)}}{\tilde{q}(x)^{1/p'(x)}}.\end{aligned}$$

Letting $Mf(x)\tilde{q}(x)^{1/p'(x)} = \delta^{-\alpha-q(x)/p(x)}$ by (6) as in the proof of Proposition 3.1.2 in [1], we find

$$\int_G |x-y|^{\alpha-n} f(y)dy \le CMf(x)^{p(x)/p^\sharp(x)}\frac{1}{\tilde{q}(x)^{\alpha(p(x)-1)/n}}.$$

Next consider the case

$$Mf(x)\tilde{q}^{1/p'(x)} \le 2^{\alpha+q(x)/p(x)}.$$

Then we have

$$\begin{aligned}\int_G |x-y|^{\alpha-n} f(y)dy &\le CMf(x) \\ &= C\left(Mf(x)\tilde{q}^{1/p'(x)}\right)\tilde{q}^{-1/p'(x)} \\ &\le C\left(Mf(x)\tilde{q}^{1/p'(x)}\right)^{p(x)/p^\sharp(x)}\tilde{q}^{-1/p'(x)} \\ &= CMf(x)^{p(x)/p^\sharp(x)}\frac{1}{\tilde{q}(x)^{\alpha(p(x)-1)/n}},\end{aligned}$$

as required. □

In view of Lemma 4.2 we see that

$$\left(\tilde{q}(x)^{\alpha(p(x)-1)/n}U_\alpha f(x)\right)^{p^\sharp(x)/p(x)} \le CMf(x)$$

for all nonnegative measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} \le 1$. Since $M$ is bounded from $L^{p(\cdot)}$ to itself according to the result by Diening [2], we have the following result.

THEOREM 4.3. *There exist positive constants $c_1$ and $c_2$ such that*

$$\int_G \left(c_1\tilde{q}(x)^{\alpha(p(x)-1)/n}U_\alpha f(x)\right)^{p^\sharp(x)} dx \le c_2$$

*for all nonnegative measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} \le 1$.*

When $\alpha = 1$, we refer the reader to the paper by Edmunds-Rákosník [4]; compare also with the paper by Diening [3] concerning Sobolev's embeddings.

REMARK 4.4. For $0 < \varepsilon < 1$, set $p(x) = n - \varepsilon$ and $1/q = 1/p(x) - 1/n$. Then we see from Lemma 4.2 that

$$(1/q)^{n/(n-1)} \|U_1 f\|_q \leq C\|f\|_{n-\varepsilon}$$

(see also [11]). Hence we have the following fact by Fusco-Lions-Sbordone [5]:

If $f$ is a nonnegative measurable function on $G$ such that

$$\lim_{\varepsilon\to 0+} \varepsilon^\delta \int_G f(y)^{n-\varepsilon} dy = 0$$

for some $0 < \delta < 1$, then

$$\int_G \exp(c(U_1 f(x))^{1/A}) dx < \infty \qquad \text{for all } c > 0,$$

where $A = (n - 1 + \delta)/n$.

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*Department of Mathematics*
*Graduate School of Science*
*Hiroshima University*
*Higashi-Hiroshima 739-8526, Japan*
*E-mail : toshi@mis.hiroshima-u.ac.jp*
*and*
*The Division of Mathematical and Information Sciences*
*Faculty of Integrated Arts and Sciences*
*Hiroshima University*
*Higashi-Hiroshima 739-8521, Japan*
*E-mail : mizuta@mis.hiroshima-u.ac.jp*