Sobolev embeddings for Riesz potential space of variable exponent

Dedicated to Prof. Makoto Ohtsuka on the occasion of his eightieth birthday

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Abstract

Our aim in this paper is to deal with Sobolev embeddings for Riesz potential spaces of variable exponent.

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We consider the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$, which is defined by

$$U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy.$$

Here $0 < \alpha < n$. Following Orlicz [15] and Kováčik and Rákosník [10], we consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ and a measurable function f satisfying

$$\int |f(y)|^{p(y)} dy < \infty.$$

In this paper we are concerned with $p(\cdot)$ satisfying the following 0-Hölder condition

$$|p(x)-p(y)| \le \frac{a_1 \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{a_2}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/2$, where a_1 and a_2 are nonnegative constants. Recently Diening [4] has established embedding results for Riesz potentials in the case $a_1 = 0$.

In these discussions, the continuity of Hardy-Littlewood maximal functions is a crucial tool. Our first task is to establish the continuity in the case $a_1 \ge 0$, which is an extension of Diening [3] in the case $a_1 = 0$. As an application of the

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continuity of maximal functions, we give Sobolev's inequality for Riesz potentials in the variable exponent case. Finally we discuss the mean continuity for Riesz potentials as extensions of Meyers [12] and Harjulehto-Hästö [8].

For related results, see Edmunds-Rákosník [5], Kováčik-Rákosník [10] and Růžička [16].

2 Maximal functions

Throughout this paper, let C denote various constants independent of the variables in question.

Let G be a bounded open set in $\mathbf{R}^n$, and consider a positive continuous function $p(\cdot)$ on G.

In this paper let us assume that :

(p1) $1 < p_-(B) = \inf_B p(x) \le \sup_B p(x) = p_+(B) < \infty \qquad$ for $B \subset G$;

(p2) $|p(x) - p(y)| \le \dfrac{a_1 \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{a_2}{\log(1/|x-y|)}$
whenever $|x-y| < 1/2$, $x \in G$ and $y \in G$.

Let $1/p'(x) = 1 - 1/p(x)$. Then, noting that

$$p'(y) - p'(x) = \frac{p(x)-p(y)}{(p(x)-1)(p(y)-1)} = \frac{p(x)-p(y)}{(p(x)-1)^2} + \frac{\{p(x)-p(y)\}^2}{(p(x)-1)^2(p(y)-1)},$$

we have the following result.

Lemma 2.1. *There exists a positive constant C such that*

$$|p'(x) - p'(y)| \le \omega(|x-y|) \qquad \text{whenever } x \in G \text{ and } y \in G,$$

where $\omega(r) = \omega(r;x) = \dfrac{a_1}{(p(x)-1)^2} \dfrac{\log(\log(1/r))}{\log(1/r)} + \dfrac{C}{\log(1/r)}$ *for* $0 < r \le r_0$ *and* $\omega(r) = \omega(r_0)$ *for* $r \ge r_0$.

For a locally integrable function f on G, we consider the maximal function Mf defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{G \cap B} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x,r)$ and $|B|$ denotes the volume of B.

Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\{\lambda > 0 : \int_G \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \le 1\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$.

LEMMA 2.2. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$Mf(x)^{p(x)} \le C\left\{Mg(x)(\log(e + Mg(x)))^{A_1(x)p(x)} + 1\right\},$$

where $g(y) = f(y)^{p(y)}$ and $A_1(x) = a_1 n/p(x)^2$.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$, and let $0 < r_0 < 1$ be fixed. First note that

$$\int_G f(y)^{p(y)} dy \le 1. \tag{1}$$

Then, if $r \ge r_0$, then

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le \frac{1}{|B(x,r)|}\int_{B(x,r)} \{1 + f(y)^{p(y)}\}dy \le C$$

by our assumption. For $0 < \mu \le 1$ and $r > 0$, we have

$$\begin{aligned} &\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \\ \le\ &\mu\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} (1/\mu)^{p'(y)}dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy\right) \\ \le\ &\mu\left((1/\mu)^{p'(x)+\omega(r)} + F\right), \end{aligned}$$

where $F = |B(x,r)|^{-1}\int_{B(x,r)} f(y)^{p(y)}dy$. Here, considering

$$\mu = F^{-1/\{p'(x)+\omega(r)\}} = F^{-1/p'(x)+\beta(x)}$$

with $\beta(x) = \omega(r)/\{p'(x)(p'(x)+\omega(r))\}$ when $F \ge 1$, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le 2F^{1/p(x)}F^{\omega(r)/p'(x)^2};$$

if $F < 1$, then we can take $\mu = 1$ to obtain

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le 2.$$

Hence it follows that

$$\frac{1}{|B(x,r))|}\int_{B(x,r)} f(y)dy \le C(F^{1/p(x)}F^{\omega(r)/p'(x)^2} + 1). \tag{2}$$

If $r \le F^{-1}$, then we see from (2) that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le C\left\{F^{1/p(x)}(\log(e+F))^{A_1(x)}+1\right\}.$$

If $r_0 > r > F^{-1}$, then

$$F^{1/p(x)+\omega(r)/p'(x)^2} \le Cr^{-n/p(x)-n\omega(r)/p'(x)^2}\left(\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)+\omega(r)/p'(x)^2}.$$

In view of (1), we find

$$\begin{aligned} F^{1/p(x)+\omega(r)/p'(x)^2} &\le Cr^{-n/p(x)}(\log(1/r))^{A_1(x)}\left(\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)+\omega(r)/p'(x)^2} \\ &\le Cr^{-n/p(x)}(\log(1/r))^{A_1(x)}\left(\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)} \\ &\le Cr^{-n/p(x)}(\log F)^{A_1(x)}\left(\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)} \\ &\le CF^{1/p(x)}(\log F)^{A_1(x)}. \end{aligned}$$

Now we have established

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le C\left\{F^{1/p(x)}(\log(e+F))^{A_1(x)}+1\right\}$$

for all $r > 0$ and $x \in G$, which completes the proof. □

REMARK 2.3. Let χ_E denote the characteristic function of E, and let

$$p(x) = p_0 - \frac{a_1\log(\log(1/|x|))}{\log(1/|x|)},$$

where $p_0 = p(0) > 1$. Consider the function $f = \chi_{D_0}$ with $D_0 = 2B_0 \setminus B_0$, where $B_0 = B(0, r_0)$ and $2B_0 = B(0, 2r_0)$. Then note :

(i) $\|f\|_{p(\cdot),D_0} \le C_1 r_0^{n/p(0)}(\log(1/r_0))^{-A_1(0)}$;

(ii) $\displaystyle\frac{1}{|B(0,r)|}\int_{B(0,r)}\left(\frac{f(x)}{\|f\|_{p(\cdot),D_0}}\right)^{p(x)}dx \le C_2 r_0^{-n}$ for $r_0 < r < 2r_0$;

(iii) $\displaystyle\left(\frac{1}{|2B_0|}\int_{2B_0}\frac{f(x)}{\|f\|_{p(\cdot),D_0}}dx\right)^{p(0)} \ge C_3 r_0^{-n}(\log(1/r_0))^{A_1(0)p(0)}$.

This means that the exponent $A_1(x)$ in Lemma 2.2 is best possible.

Let $p_0(x) = p(x)/p_0$ for $1 < p_0 < p_-(G)$. Then Lemma 2.2 yields

$$Mf(x)^{p_0(x)} \le C\left\{Mg(x)(\log(e+Mg(x)))^{\tilde{a}_1 n/p_0(x)} + 1\right\}$$

for $x \in G$, where $g(y) = f(y)^{p_0(y)}$ and $\tilde{a}_1 = a_1/p_0$.

Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = an/p(x)^2$. Then we can choose p_0 so that $a_1 p_0 \le a$ and

$$Mf(x)^{p(x)} \le C\left\{Mg(x)(\log(e+Mg(x)))^{A(x)p(x)/p_0} + 1\right\}^{p_0},$$

which yields

$$\left\{Mf(x)(\log(e+Mf(x)))^{-A(x)}\right\}^{p(x)} \le C(Mg(x)+1)^{p_0}.$$

Hence we have the following result by the continuity of maximal functions in L^{p_0}.

THEOREM 2.4. *Let $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$. Set $A(x) = an/p(x)^2$. If $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G \left\{Mf(x)(\log(e+Mf(x)))^{-A(x)}\right\}^{p(x)} dx \le C.$$

When $a_1 = 0$, Theorem 2.4 was proved by Diening [3]. For the continuity of maximal functions in general domains, see Cruz-Uribe, Fiorenza and Neugebauer [2].

REMARK 2.5. Let $p(\cdot)$ be a positive continuous function on G such that $1 \le p(x) \le p_+(G) < \infty$. Then we can prove the following weak type result for maximal functions:

$$|E_f(t)| \le C \int_G \left|\frac{f(y)}{t}\right|^{p(y)} dy$$

whenever $t > 0$ and $f \in L^{p(\cdot)}(G)$, where $E_f(t) = \{x \in G : Mf(x) \ge t\}$; for this see also Cruz-Uribe, Fiorenza and Neugebauer [2, Theorem 1.8].

To prove this, we may assume that $t = 1$. We have for $\mu > 1$

$$\begin{aligned} & \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \\ \le\ & \mu\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} (1/\mu)^{p'(y)}dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{p(y)}dy\right) \\ \le\ & \mu\left((1/\mu)^{(p_+)'} + F\right), \end{aligned}$$

where $F = |B(x,r)|^{-1}\int_{B(x,r)} |f(y)|^{p(y)}dy$. Here, considering $\mu = F^{-1/(p_+)'}$ when $F < 1$, we find

$$1 \le 2F^{1/p_+},$$

so that

$$\left(\frac{1}{2}\right)^{p_+} \le M(|f|^{p(\cdot)})(x) \qquad \text{for } x \in E_f(1),$$

which proves the required assertion.

Remark 2.6. For $0 < r < 1/2$, let

$$G = \{x = (x_1, x_2) : 0 < x_1 < 1, -1 < x_2 < 1\}$$

and

$$G(r) = \{x = (x_1, x_2) : 0 < x_1 < r, r < x_2 < 2r\}.$$

For $a_1 > 0$ and $p(0) = p_0 > 1$, define

$$p(x_1, x_2) = \begin{cases} p_0 - a_1 \log(\log(1/x_2))/\log(1/x_2) & \text{when } 0 < x_2 \le r_0, \\ p_0 & \text{when } x_2 \le 0; \end{cases}$$

set $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Here we take $r_0 > 0$ so small that $p(x_1, r_0) > 1$. Consider

$$f_r(y) = \chi_{G(r)}(y)$$

and set $g_r = f_r/\|f_r\|_{p(\cdot),G}$. Then we insist for $0 < r < r_0$:

(i) $\|f_r\|_{p(\cdot),G} \le C_1 r^{2/p(0)} (\log(1/r))^{-A_1(0)}$;

(ii) $Mg_r(x) \ge C_2 r^{-2/p(x)} (\log(1/r))^{A_1(x)}$ for $0 < x_1 < r$ and $-r < x_2 < 0$.

By integration of (ii) we see that

$$\int_G \left\{ Mg_r(x)(\log(e + Mg_r(x))^{-A_1(x)} \right\}^{p(x)} dx \ge C_3,$$

which means that Theorem 2.4 does not hold for $0 < a < a_1$.

3 Riesz potentials

For $0 < \alpha < n$, we consider the Riesz potential of $f \in L^{p(\cdot)}(G)$ defined by

$$U_\alpha f(x) = \int_G |x - y|^{\alpha - n} f(y) dy.$$

In this section, suppose $p_+(G) < n/\alpha$ and let

$$1/p^\sharp(x) = 1/p(x) - \alpha/n.$$

Lemma 3.1. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{\alpha - n} f(y) dy \le C \delta^{-n/p^\sharp(x)} \log(1/\delta)^{A_1(x)}$$

for $x \in G$ and $0 < \delta < 1/2$, where $A_1(x) = a_1 n/p(x)^2$ as before.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. For $\mu > 1$ we have

$$\begin{aligned}\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy &\le \mu \left(\int_{G\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy + \int_{G\setminus B(x,\delta)} f(y)^{p(y)}dy \right) \\ &\le \mu \left(\int_{G\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy + 1 \right).\end{aligned}$$

Note here that

$$\begin{aligned}&\int_{B(x,\mu^{1/(\alpha-n)})\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy \\ \le\ & \int_{B(x,\mu^{1/(\alpha-n)})\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(x)+\omega(|x-y|)}dy \\ \le\ & \mu^{-p'(x)-\omega(\delta)} \int_{G\setminus B(x,\delta)} |x-y|^{(\alpha-n)(p'(x)+\omega(|x-y|))}dy \\ \le\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{(\alpha-n)(p'(x)+\omega(\delta))+n} \\ \le\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{p'(x)(\alpha-n/p(x))} (\log(1/\delta))^{(n-\alpha)a_1/(p(x)-1)^2} \\ =\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{-p'(x)n/p^\sharp(x)} (\log(1/\delta))^{(n-\alpha)a_1/(p(x)-1)^2}.\end{aligned}$$

Considering $\mu = \delta^{-n/p^\sharp(x)}(\log(1/\delta))^{A_1(x)}$, we see that

$$\int_{B(x,\mu^{1/(\alpha-n)})\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)}dy \le C,$$

so that

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-n/p^\sharp(x)}(\log(1/\delta))^{A_1(x)},$$

as required. □

LEMMA 3.2. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\rho(U_\alpha f(x), A_1(x))^{p^\sharp(x)} \le C\left\{\rho(Mf(x), A_1(x))^{p(x)} + 1\right\},$$

where $\rho(t,y) = t\,(\log(e+t))^{-y}$.

PROOF. For $0 < \delta < 1/2$ we have by Lemma 3.1

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\delta^{-n/p^\sharp(x)}(\log(1/\delta))^{A_1(x)}.$$

Considering $\delta = Mf(x)^{-p(x)/n}(\log(e+Mf(x)))^{a_1/p(x)}$ when $Mf(x)$ is large enough, we see that

$$U_\alpha f(x) \le C\left\{Mf(x)^{p(x)/p^\sharp(x)}(\log(e+Mf(x)))^{a_1\alpha/p(x)} + 1\right\}.$$

Hence it follows that

$$\rho(U_\alpha f(x), A_1(x))^{p^\sharp(x)} \le C\left\{\rho(Mf(x), A_1(x))^{p(x)} + 1\right\},$$

as required. □

REMARK 3.3. Let p and $f = \chi_{D_0}$ be as in Remark 2.3. Set $g = f/\|f\|_{p(\cdot),D_0}$. Then note :

(i) $Mg(0) \le C_1 r_0^{-n/p(0)}(\log(1/r_0))^{A_1(0)}$;

(ii) $U_\alpha g(0) \ge C_2 r_0^{-n/p^\sharp(0)}(\log(1/r_0))^{A_1(0)}$.

This means that the exponent $A_1(x)$ in Lemma 3.2 is best possible.

Let $a > a_1 > 0$ or $a = a_1 = 0$. Set $A(x) = an/p(x)^2$. In view of Theorem 2.4 and Lemma 3.2 with a_1 replaced by a, we have the following result, which gives an extension of Diening [4].

THEOREM 3.4. *Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = an/p(x)^2$. Suppose $p_+(G) < n/\alpha$. If f is a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G \left\{U_\alpha f(x)(\log(e + U_\alpha f(x)))^{-A(x)}\right\}^{p^\sharp(x)} dx \le C.$$

4 Mean continuity

If $f \in L^{p_0}(G)$ with $p_0 > 1$, then we know that

$$\lim_{r\to 0+} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |U_\alpha f(x) - U_\alpha f(x_0)|^{p_0^\sharp}\, dx = 0$$

holds for all $x_0 \in G$ except in a set of capacity zero, where $1/p_0^\sharp = 1/p_0 - \alpha/n$. If this is true, then x_0 is the Lebesgue point of $U_\alpha f$; see e.g. [1], [11], [12], [13]. To extend this well-known fact to the case of variable exponent, we first prepare the notion of $L^{p(\cdot)}$-capacity.

Let G be a bounded open set in $\mathbf{R}^n$ as before. For $E \subset G$, we define the relative $(\alpha, p(\cdot))$-capacity by

$$C_{\alpha,p(\cdot)}(E; G) = \inf \int_G f(y)^{p(y)} dy,$$

where the infimum is taken over all nonnegative functions $f \in L^{p(\cdot)}(G)$ such that $U_\alpha f(x) \ge 1$ for every $x \in E$. For another Sobolev capacity, we also refer the reader to the paper by Harjulehto-Hasto-Koskenoja-Varonen [9].

From now on we collect fundamental properties for our capacity, following Meyers [11]. Let us begin with the following result, which is proved in a way similar to the case of constant exponent (see Meyers [11]).

LEMMA 4.1. *For $E \subset G$, $C_{\alpha,p(\cdot)}(E;G) = 0$ if and only if there exists a nonnegative function $f \in L^{p(\cdot)}(G)$ such that $U_\alpha f(x) = \infty$ for every $x \in E$.*

For $0 < r \le 1/2$, set

$$h(r;x) = \begin{cases} r^{n-\alpha p(x)}(\log(1/r))^{\alpha a_1} & \text{when } p(x) < n/\alpha, \\ (\log(1/r))^{\alpha(a_1-(n-\alpha)/\alpha^2)} & \text{when } p(x) = n/\alpha \text{ and } a_1 < (n-\alpha)/\alpha^2, \\ (\log(\log(1/r)))^{-a_1\alpha} & \text{when } p(x) = n/\alpha \text{ and } a_1 = (n-\alpha)/\alpha^2, \\ 1 & \text{otherwise;} \end{cases}$$

set for simplicity $h(r;x) = h(r_0,x)$ for $r > 1/2$.

LEMMA 4.2. *Suppose $p(x) \le n/\alpha$ and $a_1 \le (n-\alpha)/\alpha^2$. If $B(x_0,r) \subset G$ and $0 < r < 1/2$, then*

$$C_{\alpha,p(\cdot)}(B(x_0,r);G) \le Ch(r;x_0).$$

PROOF. If we consider the potential

$$u(x) = \int_G |x-y|^{\alpha-n} dy,$$

then we see that $C_{\alpha,p(\cdot)}(G;G) < \infty$. Hence we have only to treat the case $0 < r < r_0 < 1/2$.

First consider the case $p(x_0) < n/\alpha$. Define

$$u(x) = \int_{B(x_0,r)\setminus B(x_0,r/2)} |x-y|^{\alpha-n}|x_0-y|^{-\alpha} dy.$$

Then, since $u(x) \ge C$ for $x \in B(x_0,r)$, we have

$$\begin{aligned} C_{\alpha,p(\cdot)}(B(x_0,r);G) &\le C\int_{B(x_0,r)\setminus B(x_0,r/2)} |x_0-y|^{-\alpha p(y)} dy \\ &\le C\int_{B(x_0,r)\setminus B(x_0,r/2)} |x_0-y|^{-\alpha(p(x_0)+\omega(|x_0-y|))} dy \\ &\le Cr^{-\alpha(p(x_0)+\omega(r))+n} \\ &\le Cr^{n-\alpha p(x_0)}(\log(1/r))^{a_1\alpha}, \end{aligned}$$

where $\omega(r) = a_1 \log(\log(1/r))/\log(1/r) + a_2/\log(1/r)$.

Next suppose $p(x_0) = n/\alpha$ and $a_1 < (n-\alpha)/\alpha^2$. Consider

$$u(x) = \int_{B(x_0,\sqrt{r})\setminus B(x_0,r)} |x-y|^{\alpha-n}|x_0-y|^{-\alpha} dy.$$

Noting that $u(x) \geq C\log(1/r)$ for $x \in B(x_0, r)$, we have

$$\begin{aligned} C_{\alpha,p(\cdot)}(B(x_0,r);G) &\leq \int_{B(x_0,\sqrt{r})\setminus B(x_0,r)} (|x_0-y|^{-\alpha}/(C\log(1/r)))^{p(y)}dy \\ &\leq C(\log(1/r))^{-p(x_0)}\int_{B(x_0,\sqrt{r})\setminus B(x_0,r)} |x_0-y|^{-\alpha(p(x_0)+\omega(|x_0-y|))}dy \\ &\leq C(\log(1/r))^{-p(x_0)}(\log(1/r))^{a_1\alpha+1} \\ &\leq C(\log(1/r))^{\alpha(a_1-(n-\alpha)/\alpha^2)}. \end{aligned}$$

Finally suppose $p(x_0) = n/\alpha$ and $a_1 = (n-\alpha)/\alpha^2$. Consider

$$u(x) = \int_{B(x_0,2r_0)\setminus B(x_0,r)} |x-y|^{\alpha-n}|x_0-y|^{-\alpha}(\log(1/|x_0-y|))^{-1}dy$$

when $0 < r < r_0$. Since $u(x) \geq C\log(\log(1/r))$ for $x \in B(x_0, r)$, we find

$$\begin{aligned} & C_{\alpha,p(\cdot)}(B(x_0,r);G) \\ \leq\ & \int_{B(x_0,2r_0)\setminus B(x_0,r)} \{|x_0-y|^{-\alpha}(\log(1/|x_0-y|))^{-1}/(C\log(\log(1/r)))\}^{p(y)}dy \\ \leq\ & C(\log(\log(1/r))^{-p(x_0)}\log(\log(1/r)) \\ =\ & C(\log(\log(1/r)))^{-a_1\alpha}. \end{aligned}$$

Thus the present lemma is proved. □

REMARK 4.3. If $p_-(G) \geq n/\alpha$ and $a_1 > (n-\alpha)/\alpha^2$, then $C_{\alpha,p(\cdot)}(\{x_0\};G) > 0$ for $x_0 \in G$. In this case, if $f \in L^{p(\cdot)}(G)$, then $U_\alpha f$ is shown to be continuous in G (see [7]).

LEMMA 4.4. *If f is a nonnegative measurable function on G with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r\to 0+} h(r;x)^{-1}\int_{B(x,r)} f(y)^{p(y)}dy = 0$$

holds for all x except in a set $E \subset G$ with $C_{\alpha,p(\cdot)}(E;G) = 0$.

PROOF. For $\delta > 0$, consider the set

$$E_\delta = \{x \in G : \limsup_{r\to 0+} h(r;x)^{-1}\int_{B(x,r)} f(y)^{p(y)}dy > \delta\}.$$

It suffices to show that $C_{\alpha,p(\cdot)}(E_\delta;G) = 0$ only when $\lim_{r\to 0+} h(r;x) = 0$ for some (or all) x.

Let $0 < \varepsilon < 1/2$. For each $x \in E_\delta$, we find $0 < r(x) < \varepsilon$ such that

$$h(r(x);x)^{-1}\int_{B(x,r(x))} f(y)^{p(y)}dy > \delta.$$

By a covering lemma, there exists a disjoint family $\{B_j\}$ such that $B_j = B(x_j, r(x_j))$ and $\bigcup_j B(x_j, 5r(x_j)) \supset E_\delta$. Then we have

$$\begin{aligned} C_{\alpha,p(\cdot)}(E_\delta; G) &\leq \sum_j C_{\alpha,p(\cdot)}(B(x_j, 5r(x_j)); G) \\ &\leq C \sum_j h(r(x_j); x_j) \\ &\leq C\delta^{-1} \int_{\cup_j B_j} f(y)^{p(y)} dy. \end{aligned}$$

Noting that $|\cup_j B_j| \leq C\delta^{-1}\varepsilon^{\alpha\tilde{p}}$ for $1 < \tilde{p} < p_-(G)$, we see that

$$C_{\alpha,p(\cdot)}(E_\delta; G) = 0,$$

as required. □

Set $\varphi(r, y) = r(\log(r + c))^{-y}$. Then for each $y_0 > 0$ we can find $c > 0$ such that

$$|\varphi(s, y) - \varphi(t, y)| \leq \varphi(|s - t|, y) \tag{3}$$

whenever $s \geq 0$, $t \geq 0$ and $0 \leq y \leq y_0$.

We are now ready to give mean continuity of Riesz potentials, which give an extension of Meyers [12] and Harjulehto-Hästö [8].

THEOREM 4.5. *Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = an/p(x)^2$. Suppose $p_+(G) < n/\alpha$. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. Consider the sets*

$$E = \{x \in G : U_\alpha f(x) = \infty\}$$

and

$$E(a) = \{x \in G : \limsup_{r \to 0+} k(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)} dy > 0\},$$

where $k(r; x) = r^{n-\alpha p(x)}(\log(1/r))^{-2A(x)p(x)}$. If $x_0 \in G \setminus (E \cup E(a))$, then

$$\lim_{r \to 0+} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} \varphi(|U_\alpha f(x) - U_\alpha f(x_0)|, A(x))^{p^\sharp(x)} dx = 0.$$

PROOF. Suppose $U_\alpha f(x_0) < \infty$ and

$$\lim_{r \to 0+} k(r; x_0)^{-1} \int_{B(x_0,r)} f(y)^{p(y)} dy = 0.$$

Write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x_0,2|x-x_0|)} |x - y|^{\alpha-n} f(y) dy + \int_{G \setminus B(x_0,2|x-x_0|)} |x - y|^{\alpha-n} f(y) dy \\ &= U_1(x) + U_2(x). \end{aligned}$$

By Lebesgue's dominated convergence theorem, we see that

$$\lim_{x\to x_0} U_2(x) = U_\alpha f(x_0) < \infty.$$

Hence, in view of (3), we have only to show that

$$\lim_{r\to 0+} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |u_r(x)|^{p^\sharp(x)} dx = 0,$$

where $u_r(x) = U_\alpha f_r(x)(\log(U_\alpha f_r(x)+c))^{-A(x)}$ with $f_r = f\chi_{B(x_0,r)}$.

We apply Theorem 3.4 with $f = f_r/\|f_r\|_{p(\cdot)}$ to obtain

$$\int_{B(x_0,r)} \left\{U_\alpha(f_r(x)/\|f_r\|_{p(\cdot)})(\log(U_\alpha(f_r(x)/\|f_r\|_{p(\cdot)})+c)^{-\tilde{A}(x)}\right\}^{p^\sharp(x)} dx \le C,$$

where $\tilde{A}(x) = \tilde{a}n/p(x)^2$ for $a_1 < \tilde{a} < a$ when $a_1 > 0$ and $\tilde{A}(x) = 0$ when $a_1 = 0$. Hence, since $p^\sharp(x) \ge p^\sharp_* = np_*/(n-\alpha p_*)$ with $p_* = p_-(B(x_0,r))$, we see that

$$\begin{aligned}&\int_{B(x_0,r)} \left\{U_\alpha f_r(x)(\log(U_\alpha f_r(x)+c))^{-A(x)}\right\}^{p^\sharp(x)} dx \\ \le\ & C\left\{\|f_r\|_{p(\cdot)}(\log(\|f_r\|^{-1}_{p(\cdot)}+c))^{A(x_0)}\right\}^{p^\sharp_*}.\end{aligned}$$

Further, since $\|f_r\|^{p^*}_{p(\cdot)} \le \int_{B(x_0,r)} f(y)^{p(y)}dy = F(r)$, $p^* = p_+(B(x_0,r))$, we find

$$\int_{B(x_0,r)} u_r(x)^{p^\sharp(x)}dx \le CF(r)^{p^\sharp_*/p^*}(\log(F(r)^{-1/p^*}+c)^{p^\sharp_* A(x_0)}.$$

If we set $\varepsilon(r) = k(r;x_0)^{-1}\int_{B(x_0,r)} f(y)^{p(y)}dy$, then we establish

$$\begin{aligned}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u_r(x)^{p^\sharp(x)}dx &\le Cr^{-n}(k(r;x_0)\varepsilon(r))^{p^\sharp_*/p^*}(\log(k(r;x_0)\varepsilon(r))^{-1})^{p^\sharp_* A(x_0)} \\ &\le C\varepsilon(r)^{p^\sharp_*/p^*}\log(1/\varepsilon(r))^{p^\sharp_* A(x_0)},\end{aligned}$$

because $r^{(n-\alpha p(x_0))p^\sharp_*/p^*} \le Cr^{(n-\alpha p(x_0))p^\sharp(x_0)/p(x_0)}(\log(1/r))^{A(x_0)p^\sharp(x_0)}$ for small r. This shows that the left hand side tends to zero as $r\to 0+$, and thus the proof is completed. □

The case $a_1 = 0$ is simple and can be stated in the following (see Harjulehto-Hästö [8]).

COROLLARY 4.6. *Suppose $a_1 = 0$ and $p_+(G) < n/\alpha$. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\lim_{r\to 0+} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |U_\alpha f(x) - U_\alpha f(x_0)|^{p^\sharp(x)}dx = 0$$

for all $x_0 \in G$ except in a set E with $C_{\alpha,p(\cdot)}(E;G) = 0$.

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